SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


Solve the equation

\[ 3^{1-x} + 3\sqrt{3x - 2x^2} = 4. \]

Solved by M. Bataille; D. M. Bătinețu-Giurgiu, N. Stanciu and T. Zvonaru; B. D. Beasley; P. Deiermann; N. Hodžić and S. Malikić; O. Kouba; D. Koukakis; C. R. Pranesachar; D. Smith; and the proposer. There were five solutions that were either incorrect or incomplete. We present the solution composed from solutions by several solvers.

Any solution must satisfy \(0 \leq x \leq 3/2.\) Two solutions are \(x = 0\) and \(x = 1.\) Note that when \(1 < x < 3/2,\) then

\[ 1 - (3x - 2x^2) = (x - 1)(2x - 1) > 0, \]

so that

\[ 3^{1-x} + 3\sqrt{3x - 2x^2} < 1 + 3 = 4. \]

Suppose that \(4/19 < x < 1.\) Then \(9(3x - 2x^2) - (2 + x)^2 = (19x - 4)(1 - x) > 0\) and so

\[ 1 - x + 3(\sqrt{3x - 2x^2} - 1) = 3\sqrt{3x - 2x^2} - (2 + x) > 0. \]

Hence, by the arithmetic-geometric means inequality, we have

\[
3^{1-x} + 3\sqrt{3x - 2x^2} = 3^{1-x} + 3\sqrt{3x - 2x^2 - 1} + 3\sqrt{3x - 2x^2 - 1} + 3\sqrt{3x - 2x^2 - 1} \\
\geq 4 \left(3^{1-x} + 3(\sqrt{3x - 2x^2 - 1})\right)^{1/4} > 4.
\]

Now, suppose that \(0 < x < 3/11.\) Then \((3x - 2x^2) - 9x^2 = x(3 - 11x) > 0,\) so that

\[
3^{1-x} + 3\sqrt{3x - 2x^2} = 3^{-x} + 3^{-x} + 3^{-x} + 3\sqrt{3x - 2x^2} \\
\geq 4 \left(3^{-3x} + 3\sqrt{3x - 2x^2}\right)^{1/4} > 4.
\]

Since \(4/19 < 3/11,\) we conclude that the equation has no solution in the set \((0, 1) \cup (1, 3/2),\) and so \(x = 0\) and \(x = 1\) are the only solutions.

Editor’s Comments. There are quick arguments for some parts of the domain. Since \(3x - 2x^2 > 1\) for \(1/2 < x < 1,\) it is easy to see that the left side of the equation exceeds 4 on this interval. Since \(3^{1-x} + 3\sqrt{3x - 2x^2}\) strictly decreases for \(3/4 \leq x \leq 3/2,\) the only solution in this interval is \(x = 1.\)

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Deiermann showed more generally that $x = 0$ and $x = 1$ are the only solutions of
\[ b\sqrt{b-a(1-x)} + b\sqrt{b-a}ax^2 = 1 + b\sqrt{b-a} \]
when $0 < a < b \leq 2a$, $1 < b$ and $c - c^{-1} \leq b(b-a)^{-1}$, where $c = b\sqrt{b-a}$. He did this by recasting the equation as $h(x) = g(x)$, with $h(x) = \sqrt{bx - ax^2\ln b}$ and $g(x) = \ln(1 + b\sqrt{b-a} - b\sqrt{b-a(1-x)})$. Noting that $h(0) = g(0) = h(b/a) = 0$ and $h(1) = g(1) = \sqrt{b-a}\ln b$, he analyzed the graphs of these two functions to show that they crossed only when $x = 0$ and $x = 1$. The above problem is the case $(a,b) = (2,3)$.


For $n \in \mathbb{N}$, let $S = \{1, 2, 3, \ldots, n\}$. For each nonempty $T \subseteq S$ define the “drop” of $T$ by $d(T) = f(T) - g(T)$ where $f(T)$ and $g(T)$ denote the maximum and minimum elements of $T$, respectively. (e.g., $d(\{2\}) = 0$, $d(\{2,3,7\}) = 5$) Evaluate $D_n = \sum d(T)$, the total of the drops of $S$, where the summation is over all non-empty subsets $T$ of $S$.

Solved by AN-anduud Problem Solving Group; M. Bataille; D. Bătinețu-Giurgiu, N. Stanciu and T. Zvonaru; P. De; O. Kouba; K. Lau; S. Malikić; Missouri State University Problem Solving Group; C. R. Pranesachar; D. Smith; E. Suppa; I. Uchiha; and the proposers. We present the solution by Itachi Uchiha slightly expanded by the editor.

Note first that each $i \in S$ is the maximum element of $2^{i-1}$ subsets of $S$ and the minimum element of $2^{n-i}$ subsets of $S$. Hence,
\[ D_n = \sum d(T) = \sum f(T) - \sum g(T) = \sum_{i=1}^{n} i2^{i-1} - \sum_{i=1}^{n} i2^{n-i} \]
\[ = \sum_{i=1}^{n} i2^{i-1} - \sum_{i=0}^{n-1} (n-i)2^i = 3\sum_{i=1}^{n} i2^{i-1} - n\sum_{i=0}^{n} 2^i. \]

For $x \neq 1$, define
\[ h(x) = \frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n. \] (1)

Now we can write
\[ D_n = 3h'(2) - nh(2). \] (2)

Since $h'(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}$ and $xh'(x) = x + 2x^2 + 3x^3 + \cdots + nx^n$, we have
\[ (1-x)h'(x) = 1 + x + x^2 + \cdots + (n-1)x^{n-1} - nx^n. \]

Or, in other words,
\[ h'(x) = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}. \] (3)

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Now, by substituting (1) and (3) into (2), we conclude that
\[ D_n = 3(1 - 2^n + n2^n) + n(1 - 2^{n+1}) \]
\[ = (n - 3)2^n + n + 3. \]

Editor’s Comment. Using a counting argument, Lau established the formula \( D_n = \sum_{k=1}^{n-1} k(n-k)2^{k-1} \). AN-anduud Problem Solving Group obtained the recurrence formula \( D_n = D_{n-1} + (n - 2)2^{n-1} + 1 \). Prithwijit gave the recurrence \( D_n = D_{n-1} + \sum_{k=1}^{n-1} k2^{k-1} \) without proof.


Let \( a, b, c \) be positive real numbers. Prove that
\[ (3a^2 + 2) \frac{a^3 + b^3}{a^2 + ab + b^2} + (3b^2 + 2) \frac{b^3 + c^3}{b^2 + bc + c^2} + (3c^2 + 2) \frac{c^3 + a^3}{c^2 + ca + a^2} \geq 10abc. \]

Solved by A. Alt; AN-anduud Problem Solving Group; Š. Arslanagić; D. Bailey, E. Campbell, and C. Diminnie; M. Bătineţu-Giurgiu; N. Staniciu and T. Zvonaru (2 solutions); P. De; C. M. Quang; N. Hodžić; O. Kouba; S. Malikić; P. Perfetti; C. R. Pranesachar; D. Smith; D. Văcaru; S. Wagon; and the proposer. We present the solution by AN-anduud Problem Solving Group.

Let \( L \) denote the left side of the given inequality. Since
\[ 3(x^2 - xy + y^2) - (x^2 + xy + y^2) = 2(x^2 - 2xy + y^2) = 2(x - y)^2 \geq 0, \]
we have
\[ \frac{x^2 - xy + y^2}{x^2 + xy + y^2} \geq \frac{1}{3}. \]

Using (1), the condition that \( ab + bc + ca = 3 \) and the AM-GM Inequality, we have that
\[ L = \sum_{\text{cyclic}} (3a^2 + 2)(a + b) \left( \frac{a^2 - ab + b^2}{a^2 + ab + b^2} \right) \]
\[ \geq \frac{1}{3} \sum_{\text{cyclic}} (3a^2 + 2)(a + b) \]
\[ = \sum_{\text{cyclic}} a^3 + \sum_{\text{cyclic}} a^2b + \frac{4}{3} \left( \sum_{\text{cyclic}} a \right) \left( \frac{ab + bc + ca}{3} \right) \]
\[ \geq 3 \sqrt[3]{a^3b^3c^3} + 3 \sqrt[3]{(a^2b)(b^2c)(c^2a)} + 4 \sqrt[3]{abc} \cdot \sqrt[3]{(ab)(bc)(ca)} \]
\[ = 3abc + 3abc + 4abc \]
\[ = 10abc. \]

Editor’s Comment. Almost all the submitted solutions are similar to the one featured above. Both Alt and Arslanagić gave the counterexample \( a = b = c = 2 \) to

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disprove the original incorrect version and both gave a variant, with proof, of the original inequality by replacing the left side with

\[ \sum (3a^2 + 2b^2) \left( \frac{a^3 + b^3}{a^2 - ab + b^2} \right). \]

Wagon’s proof was based on using Mathematica’s FindInstance.

Arslanagić, Bailey, Campbell, Diminnie, Bătinețu-Giurgiu, Stanciu, and Zvonaru all gave another variant, with proof, in which the condition is \( a + b + c = 3 \). In addition, Kouba gave a variant, with proof, in which the condition is \( abc = 1 \). Bătinețu-Giurgiu, Stanciu, and Zvonaru gave the following two generalizations:

1. If \( a, b, c, m, n \in (0, \infty) \) such that \( ab + bc + ca = 3 \), then

\[ \sum (ma^2 + n) \left( \frac{a^3 + b^3}{a^2 + ab + b^2} \right) \geq 2(m + n)abc. \]

2. If \( a, b, c, m, n, k \in (0, \infty) \) such that \( ab + bc + ca \leq k \), then

\[ \sum (ma^2 + n) \left( \frac{a^3 + b^3}{a^2 + ab + b^2} \right) \geq \left( \frac{2(mk + 3n)}{k} \right) abc. \]


Let \( ABC \) be a triangle with circumradius \( R \), inradius \( r \) and semiperimeter \( s \) for which we denote \( Q = \sum \text{cyclic} \cos \left( \frac{A}{2} \right) \). Prove that

\[ s = 2Q \left( \sqrt{R^2Q^2 - Rr - 2R} \right). \]

Solved by A. Alt; M. Bataille; K. Lau; S. Malkić; C. R. Pranesachar; P. Y. Woo; T. Zvonaru; and the proposer. We present the solution by Kee-Wai Lau, modified slightly by the editor.

The following well-known identities can be found as entries 56, 57, and 58 in Recent Advances in Geometric Inequalities by D.S. Mitrinović, J.E. Pečarić, and V. Volenc (Kluwer Academic Publishers, The Netherlands, 1989):

\[ \sum \text{cyclic} \cos^2 \left( \frac{A}{2} \right) = \frac{4R + r}{2R}, \]
\[ \sum \text{cyclic} \cos^2 \left( \frac{A}{2} \right) \cos^2 \left( \frac{B}{2} \right) = \frac{s^2 + (4R + r)^2}{16R^2}, \]
\[ \cos \left( \frac{A}{2} \right) \cos \left( \frac{B}{2} \right) \cos \left( \frac{C}{2} \right) = \frac{s}{4R}. \]
Hence,

\[ Q^2 = \left( \sum_{cyclic} \cos \left( \frac{A}{2} \right) \right)^2 = \frac{4R + 2}{2R} + 2 \sum_{cyclic} \cos \left( \frac{A}{2} \right) \cos \left( \frac{B}{2} \right) \]  

(1)

and

\[ (2RQ^2 - (4R + r))^2 = \left( 4R \sum_{cyclic} \cos \left( \frac{A}{2} \right) \cos \left( \frac{B}{2} \right) \right)^2 \]

\[ = 16R^2 \left( \sum_{cyclic} \cos \left( \frac{A}{2} \right) \cos \left( \frac{B}{2} \right) \right)^2 \]

\[ = 16R^2 \left( \frac{s^2 + (4R + r)^2}{16R^2} + 2 \left( \frac{s}{4R} \right) Q \right) \]

\[ = s^2 + (4R + r)^2 + 8RQs \]  

(2)

Simplifying (2), we obtain

\[ 4R^2Q^4 - 4RQ^2(4R + r) = s^2 + 8RQs \quad \text{or} \]

\[ s^2 + 8RQs + 4RQ^2(4R + r) - 4R^2Q^4 = 0. \]  

(3)

The discriminant of the quadratic equation \( f(s) = 0 \) in (3) is

\[ D = 64R^2Q^2 - 16RQ^2(4R + r) + 16R^2Q^4 \]

\[ = 16R^2Q^4 - 16rRQ^2 \]

\[ = 16R^2(Q^2 - r) \]  

(4)

From (1) it is clear that \( Q^2 > 2 \). Furthermore, \( R \geq 2r \) by Euler’s formula. Hence, from (4), \( RQ^2 - r > 2R - r > 0 \), so \( D > 0 \). Therefore, \( f(s) \) has two real roots given by

\[ s = \frac{1}{2} \left( -8RQ \pm \sqrt{16R^2(Q^2 - r)} \right) = 2Q \left( \pm \sqrt{R^2Q^2 - Rr - 2R} \right). \]

Rejecting the negative root, we finally have \( s = 2Q \left( \sqrt{R^2Q^2 - Rr - 2R} \right) \), which completes the proof.


Consider an ellipse \( E \) given by the equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) with \( a > b \). Find the coordinates, in the first quadrant, of the point \( P \) on \( E \) such that the acute angle \( \theta \) between the tangent \( t \) to \( E \) at \( P \) and the line \( OP \) is minimized.
Let \((x, y)\) be the coordinates of a variable point \(P\) of the ellipse in the first quadrant; that is \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) and \(x, y > 0\). The line \(OP\) has slope \(\frac{y}{x}\) while the tangent to the ellipse at \(P\) has slope \(-\frac{b^2}{a^2}\), so the acute angle between these lines satisfies (with the help of the AM-GM inequality)

\[
\tan \theta = \frac{y}{x} - \left( -\frac{b^2}{a^2} \right) = \frac{1}{a^2 - b^2} \left( \frac{a^2 y}{x} + \frac{b^2 x}{y} \right) \geq \frac{2ab}{a^2 - b^2}.
\]

Equality holds if and only if \(\frac{a^2 y}{x} = \frac{b^2 x}{y}\); that is, if and only if \(\frac{a^2}{x^2} = \frac{b^2}{y^2}\). But in the equation of the ellipse these equal fractions sum to 1, so they must each equal \(\frac{1}{2}\). Since the tangent function is strictly increasing, \((\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})\) is the point where the acute angle \(\theta\) achieves its minimum.


Let \(ABC\) be a triangle with medians \(m_a, m_b\) and \(m_c\), circumradius \(R\) and inradius \(r\). Let \(P\) be the point of intersection of the bisector of \(\angle A\) and the median from \(B\), \(Q\) be the point of intersection of the bisector of \(\angle B\) and the median from \(C\), and \(R\) be the point of intersection of the bisector of \(\angle C\) and the median from \(A\). If \(\angle APB = \alpha\), \(\angle BQC = \beta\) and \(\angle CRA = \gamma\), prove that

\[
\frac{m_a m_b m_c \sin \alpha \sin \beta \sin \gamma}{(a + 2b)(b + 2c)(c + 2a)} = \frac{r}{32R}.
\]

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Similarly,
\[
\frac{m_c \sin \beta}{c + 2a} = \frac{1}{2} \sin \frac{B}{2} \quad \text{and} \quad \frac{m_a \sin \gamma}{a + 2b} = \frac{1}{2} \sin \frac{C}{2}.
\]
By multiplication it follows that
\[
\frac{m_am_bm_c \sin \alpha \sin \beta \sin \gamma}{(a + 2b)(b + 2c)(c + 2a)} = \frac{1}{8} \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.
\]
The desired result follows immediately from the identity
\[
\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{4R},
\]
which holds for all triangles \(ABC\).


Let \(S\) be a finite set with cardinality \(|S| = n \geq 1\) and let \(k\) be a positive integer. Calculate
\[
\sum_{(A)} |A(1) \cap A(2) \cap \cdots \cap A(k)| \quad \text{and} \quad \sum_{(A)} |A(1) \cup A(2) \cup \cdots \cup A(k)|,
\]
where the summation \(\sum_{(A)}\) is over all mappings \(A\) from \(\{1, 2, \ldots, k\}\) to the power set \(P(S)\).

**Solved by AN-anduud Problem Solving Group; O. Geupel; O. Kouba; Missouri State University Problem Solving Group; C. R. Pranesachar; and the proposer. Skidmore College Problem Group provided correct solution but without proof. We present the solution by Oliver Geupel.**

Let us introduce convenient notation:
\[
f(n, k) = \sum_{(A)} |A(1) \cap A(2) \cap \cdots \cap A(k)|,
\]
\[
g(n, k) = \sum_{(A)} |A(1) \cup A(2) \cup \cdots \cup A(k)|.
\]
We prove that
\[
f(n, k) = n \cdot 2^{k(n-1)}, \quad (1)
\]
\[
g(n, k) = n \cdot 2^{k(n-1)}(2^k - 1). \quad (2)
\]
Without loss of generality, we may assume that \(S = \{1, 2, \ldots, n\}\). We let \(\mathcal{F}\) denote the set of all mappings \(A : \{1, 2, \ldots, k\} \rightarrow P(S)\). Then for any \(A \in \mathcal{F}\), we define a \(k \times n\) matrix \(M(A) = (a_{ij})\) such that
\[
a_{ij} = \begin{cases} 0 & \text{if } j \notin A(i), \\ 1 & \text{if } j \in A(i). \end{cases}
\]

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Then it is easy to check that this correspondence defines a bijection between $\mathcal{F}$ and the set of all $k \times n (0,1)$-matrices.

Note that for any $j \in S$, we have $j \in A(1) \cap A(2) \cap \cdots \cap A(k)$ if and only if the $j$-th column of the matrix $M(A)$ is $(1,1,\ldots,1)^T$. This holds for $2^{k(n-1)} (0,1)$-matrices since the entries outside of the $j$-th column can be chosen arbitrarily from $\{0,1\}$. Hence, every element of $S$ contributes a portion $2^{k(n-1)}$ to $f(n,k)$ and (1) follows.

Next, for any $j \in S$, we have $j \in A(1) \cup A(2) \cup \cdots \cup A(k)$ if and only if the $j$-th column of $M(A)$ is distinct from $(0,0,\ldots,0)^T$. This holds for $2^{k(n-1)}(2^k - 1)$ $(0,1)$-matrices since the entries outside of the $j$-th column can be chosen arbitrarily from $\{0,1\}$ and exactly one of the $2^k$ choices for the $j$-th column, namely $(0,0,\ldots,0)^T$, is forbidden. Thus, every element of $S$ contributes a portion $2^{k(n-1)}(2^k - 1)$ to $g(n,k)$ and (2) follows.


Let $a$, $b$ and $c$ be the sides of an acute-angled triangle $ABC$. Let $H$ be the orthocentre, and let $d_a$, $d_b$ and $d_c$ be the distances from $H$ to the sides $BC$, $CA$ and $AB$, respectively. Prove that

$$\sum_{\text{cyclic}} \sqrt{\frac{1}{a^2 b^2} + \frac{1}{b^2 c^2} - \frac{1}{c^2 a^2}} \leq \frac{9}{4(d_a + d_b + d_c)^2}.$$ 

Solved by A. Alt; G. Apostolopoulos; Š. Arslanagić; M. Bataille; J. G. Heuver; N. Hodžić and Malikić; and the proposer. We present the solution by John G. Heuver modified by the editor.

Let $s$, $r$, and $R$ denote the semiperimeter, the inradius, and the circumradius of $\triangle ABC$, respectively. Furthermore, let $L$ denote the summation of the left side of the given inequality, and set $\alpha = \angle A$, $\beta = \angle B$, and $\gamma = \angle C$ for notational convenience.

We shall make use of the following identities or inequalities all of which are either well-known or easy to show:

(a) $c \cos \beta + b \cos \gamma = a$

(b) $abc = 4rsR$

(c) $d_c = 2R \cos \beta \cos \gamma$

(d) $\sum \cos \beta \cos \gamma = \frac{s^2 + r^2 - 4R^2}{4R^2}$

(e) $s^2 \leq 4R^2 + 4rR + 3r^2$ (Gerretsen’s Inequality)

(f) $2r \leq R$ (Euler’s Inequality)

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By the Root-Mean-Square Inequality, together with (a) and (b), we have

\[ L = \sum_{cyclic} \sqrt{\frac{c^2 + a^2 - b^2}{a^2b^2c^2}} = \frac{1}{abc} \sum_{cyclic} \sqrt{2ca \cos \beta} \]

\[ = \frac{1}{abc} \sum_{cyclic} \sqrt{2ca \cos \beta + \sqrt{2ab \cos \gamma}} \]

\[ \leq \frac{1}{abc} \sqrt{a(c \cos \beta + b \cos \gamma)} = \frac{1}{abc} \sum_{cyclic} \sqrt{a^2} \]

\[ = \frac{2s}{abc} = \frac{2s}{4rsR} = \frac{1}{2rR}. \]

\[ (1) \]

Next, by (c) and (d), we have

\[ \sum_{cyclic} d_c = 2R \sum_{cyclic} \cos \beta \cos \gamma = 2R \left( \frac{s^2 + r^2 - 4R^2}{4R^2} \right) = \frac{s^2 + r^2 - 4R^2}{2R}. \]

\[ (2) \]

We now show that

\[ \frac{1}{2rR} \leq \frac{9R^2}{(s^2 + r^2 - 4R^2)^2} \]

or equivalently that

\[ (s^2 + r^2 - 4R^2)^2 \leq 18rR^3. \]

\[ (3) \]

\[ (4) \]

By (e) and (f), we have

\[ (s^2 + r^2 - 4R^2)^2 \leq (r^2 + 4rR + 3r^2)^2 = 16r^2(r + R)^2 \leq 16r \left( \frac{R}{2} \right) \left( \frac{3R}{2} \right)^2 = 18rR^3. \]

So (4) holds and (3) follows.

Finally, from (1), (2), and (3), we have

\[ L \leq \frac{9R^2}{(s^2 + r^2 - 4R^2)^2} = \frac{9}{4} \left( \frac{2R}{s^2 + r^2 - 4R^2} \right)^2 \]

\[ = \frac{9}{4} \left( \frac{d_c + d_b + d_a}{2} \right)^2, \]

and our proof is complete.

\[ 3789. \quad \text{[2012 : 379, 381] Proposed by Michel Bataille.} \]

Let triangle \( ABC \) be inscribed in a circle with centre \( O \) and radius \( R \) and \( P \) be any point in its plane. Let \( P' \) be such that \( \Delta PBP' \) is directly similar to \( \Delta COA \) and \( P'' \) be the reflection of \( P \) in \( AC \). Prove that

\[ P'P'' \geq \frac{2F}{R} \]

where \( F \) is the area of \( \Delta ABC \). For which \( P \) does equality hold?

\textit{Crux Mathematicorum, Vol. 39(9), November 2013}
Solved by O. Geupel; L. Giugiuc; O. Kouba; C. R. Pranesachar; and the proposer. We present four different solutions.

Solution 1 by the proposer.

Let $\rho_{MN}$ denote the reflection whose axis is the line $MN$. Since $BP = BP'$ and $\angle PBP' = \angle COA = 2\angle CBA$, it follows that $P = \rho_{BC} \circ \rho_{AB}(P')$. As a result, $P'' = g(P')$ where $g$ is the glide reflection $\rho_{AC} \circ \rho_{BC} \circ \rho_{AB}$. Let us determine $g$.

Let $H_1, H_2, H_3$ be the feet of the altitudes from $A, B, C$, respectively and let $C_1 = \rho_{AB}(C), B_1 = \rho_{AC}(B), U = \rho_{AB}(H_1), V = \rho_{AC}(H_1)$. Since $g(C_1) = C$ and $g(B) = B_1$, the axis of $g$ passes through the respective midpoints $H_3$ and $H_2$ of $C_1C$ and $BB_1$. Thus, the axis is the line $H_3H_2$ when $\angle BAC \neq 90^\circ$. The homothety with centre $A$ that transforms $H_1$ into the orthocentre $H$ of triangle $ABC$ also transforms the midpoints $K, L$ of $H_1U, H_1V$ to $H_3, H_2$ respectively. Thus, $KL\parallel H_3H_2$; since $UV\parallel KL$, we have that $UV\parallel H_3H_2$. But $g(U) = V$, so the midpoint of $UV$ is on the axis $H_3H_2$ of $G$. Therefore $U, V, H_2, H_3$ are collinear and so $g = \rho_{UV} \circ \tau_{UV} = \tau_{UV} \circ \rho_{UV}$, where $\tau_{UV}$ denotes the translation that takes $U$ to $V$. (This decomposition remains valid when $\angle BAC = 90^\circ$.)

Since $P'' = g(P')$, we see $P''P'' \geq UV$ with equality if and only if $P'$ and $P''$ are on the axis $UV$ of $g$. The calculation of $UV$ is straightforward. Since $A, K, H_1, L$ are on the circle with diameter $AH_1$, we have $KL = AH_1 \cdot \sin A$, and so

$$UV = 2KL = 2 \sin A \cdot \frac{2F}{BC} = \frac{2F}{R}.$$

The desired inequality follows, with equality if and only if $P$ lies on the line $H_1H_2$, the reflection of $UV$ in $AC$. Note that the line through $H_1$ is perpendicular to $OC$.

Solution 2 by Leonard Giugiuc.

We set the situation in the complex plane with the affixes $A(2a), B(2i), C(-2c)$ with $a$ and $c$ real and $a + c > 0$. Let $D(0)$, the foot of the altitude from $A$, be at the origin. The centre $O$ of the circle is located at $(a - c) + (1 - ac)i$, as can be seen by computing its distances from $A, B$ and $C$. Let $P(z)$, with $z = 2x + 2yi$, and $P'(w)$ be two vertices of triangle $PBP'$. Since the triangles $COA$ and $PBP'$ are directly similar

$$\frac{A - O}{C - O} = \frac{P' - B}{P - B},$$

so that

$$\frac{a + c + (ac - 1)i}{-(a + c) + (ac - 1)i} = \frac{w - 2i}{2[x + (y - 1)i]}.\$$

Let $u = a + c$ and $v = ac - 1$. Then $u^2 + v^2 = R^2$, $2F = 4u$ and

$$w = 2i - 2[x + (y - 1)i](u + vi)^2(u^2 + v^2)^{-1}.$$

Noting that the affix of $P''$ is $\bar{z}$, we compute

$$\bar{z} - w = 2[x - (y + 1)i] + 2[(u^2 + v^2)^{-1}(u + vi)^2(x + (y - 1)i)]$$

$$= 4(u^2 + v^2)^{-1}[xu^2 - (y - 1)uv + (-yu^2 - u^2 + xuv)i].$$

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The desired inequality is equivalent to \((u^2 + v^2)|\bar{z} - w|^2 \geq (4u)^2\) or
\[
[x^2u^4 + (y - 1)^2u^2v^2 - 2x(y - 1)u^3v] + [y^2v^4 + u^4 + x^2u^2v^2 + 2yuv^2v^2 - 2xu^3v - 2xyuv^3] \\
\geq u^4 + u^2v^2,
\]
which reduces to \((xu - yv)^2(u^2 + v^2) \geq 0\). Thus, the inequality holds with equality occurring if and only if \(xu = yv\), \textit{i.e.}, \(P\) is on the line that contains \(D\) and is perpendicular to \(CO\).

\textit{Solution 3 by Omran Kouba.}

We situate the problem in the complex plane with \(O\) at the origin, \(R = 1\) and the vertices \(A, B, C\) at the respective points \(a, b, 1\).

Suppose that \(z, z'\) and \(z''\) are the respective affixes of \(P, P'\) and \(P''\). Then \(z' = b + a(z - b)\) and \(z'' = 1 + a - a\bar{z}\). (Note that \(|z'' - 1| = |z - 1|\) and \(|z'' - a| = |z - a|\), so that \(AC\) right bisects \(PP''\).)

Since
\[
z'' - z' = 1 + a - b + ab - a(z + \bar{z}) \\
= a(\bar{a} + 1 - \bar{a}b + b - 2\text{Re }z),
\]
we have that
\[
P'P'' = 2|v - \text{Re }z| = 2\sqrt{(\text{Im } v)^2 + |\text{Re } v - \text{Re } z|^2},
\]
where \(v = \frac{1}{2}(1 + \bar{a} + b - \bar{a}b)\). Thus
\[
P'P'' \geq 2|\text{Im } v|
\]
with equality if and only if \(\text{Re } z = \text{Re } v\).

We now interpret \(\text{Im } v\) and \(\text{Re } v\). Recalling the formula \(\frac{1}{2}\text{Im } (\bar{z}_1z_2 + \bar{z}_2z_3 + \bar{z}_3z_1)\) for the area of a triangle with vertices at \(z_1, z_2, z_3\), we find that
\[
F = \frac{1}{2}|\text{Im } (\bar{a} + \bar{a}b + b)| = \frac{1}{2}|\text{Im } (\bar{a} - \bar{a}b + b)| = |\text{Im } v|.
\]

The image of the point \(B\) under reflection in \(AC\) has the affix \(b'' = 1 + a - \bar{a}b\), so that \(\frac{1}{2}(b'' + b) = \bar{v} + \frac{1}{2}(\bar{b} - \bar{b})\) is the affix of the midpoint between \(B\) and its reflection \(AC\), \textit{i.e.}, the foot \(D\) of the perpendicular from \(B\) to \(AC\). Thus the equation \(\text{Re } z = \text{Re } v = \text{Re } \bar{v}\) represents the line perpendicular to \(OC\) that passes through \(D\). (One way to find the reflected image of \(B\) is to perform the rotation \(z \to 1 + (z - 1)e^{i\theta}\) where \(a = -e^{-2i\theta}\); this rotates \(CA\) onto the real axis. Reflect the image of \(B\) in the real axis and rotate back.)

In conclusion, we have proved that \(P'P'' \geq 2F\) with equality if and only if \(P\) belongs to the line through the foot of the altitude to \(AC\) that is orthogonal to \(OC\). Since \(R = 1\), this solves the problem.

\textit{Crux Mathematicorum, Vol. 39(9), November 2013}
Solution 4 by Oliver Geupel.

We use Cartesian coordinates. Wolog, let the circumcircle of triangle \(ABC\) be the unit circle and let \(AC\) be parallel to the \(y\)-axis, so that we have \(A = (\cos \alpha, \sin \alpha)\), \(B = (\cos \beta, \sin \beta)\) and \(C = (\cos \alpha, -\sin \alpha)\). The area \(F\) of the triangle is equal to \(|\sin \alpha (\cos \alpha - \cos \beta)|\).

Let \(D, Q\) and \(S\) be, respectively, the midpoint of \(AC\), the midpoint of \(PP'\) and the foot of the perpendicular from \(P\) to \(AC\). Then \(P'P'' = 2QS\). The triangle \(PBQ\) is obtained from the triangle \(COD\) by the following successive transformations:

(i) Scale triangle \(COD\) by a factor \(r\) to obtain triangle \(C_1OD_1\);
(ii) Rotate triangle \(C_1OD_1\) around \(O\) by an angle \(\phi\) to obtain triangle \(C_2OD_2\);
(iii) Translate \(C_2OD_2\) by the vector \(\overrightarrow{OB} = (\cos \beta, \sin \beta)\) to obtain triangle \(PBQ\).

We compute the coordinates of various points step by step:

\[
D = (\cos \alpha, 0),
\]
\[
C_1 = (r \cos \alpha, -r \sin \alpha),
\]
\[
D_1 = (r \cos \alpha, 0),
\]
\[
C_2 = (r \cos \alpha \cos \phi + r \sin \alpha \sin \phi, r \cos \alpha \sin \phi - r \sin \alpha \cos \phi),
\]
\[
D_2 = (r \cos \alpha \cos \phi, r \cos \alpha \sin \phi),
\]
\[
P = (r \cos \phi + \sin \alpha \sin \phi + \cos \beta, r \cos \cos \phi - \sin \alpha \cos \phi + \sin \beta),
\]
\[
Q = (r \cos \alpha \cos \phi + \cos \beta, r \cos \alpha \sin \phi + \sin \beta),
\]
\[
S = (\cos \alpha, r \cos \alpha \sin \phi - \sin \alpha \cos \phi + \sin \beta).
\]

Therefore

\[
(P'P'')^2 = 4QS^2
\]
\[
= 4(r \cos \phi - \cos \alpha (\cos \alpha - \cos \beta))^2 + 4 \sin^2 \alpha (\cos \alpha - \cos \beta)^2
\]
\[
= 4(r \cos \phi - \cos \alpha (\cos \alpha - \cos \beta))^2 + \left(\frac{2F}{R}\right)^2 \geq \left(\frac{2F}{R}\right)^2.
\]

This proves the required inequality. Equality occurs if and only if

\[
r \cos \phi = \cos \alpha (\cos \alpha - \cos \beta),
\]

that is when \(P\) lies on a line perpendicular to \(OC\).

Editor’s Comment. One solver erroneously claimed, without explanation, that the result failed to hold when \(A = P = P''\). However, if we have an equilateral triangle inscribed in a unit circle, then \(2F = 3 \times (\sqrt{3}/2) \leq 3 = P'P''\).
Let $a, \alpha \geq 0$ be nonnegative real numbers and let $\beta$ be a positive number. Determine the limit

$$L(\alpha, \beta) = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k^\alpha}{(n^2 + kn + a)^\beta}.$$ 

_Solved by M. Bataille; O. Geupel; R. Hess; A. Kotronis; O. Kouba; P. Perfetti; and the proposer. We present the solution by Oliver Geupel._

We prove that

$$L(\alpha, \beta) = \begin{cases} 
0 & \text{if } \alpha + 1 < 2\beta, \\
\infty & \text{if } \alpha + 1 > 2\beta, \\
\int_0^1 \frac{x^{2\beta-1}}{(1+x)^\beta} \, dx & \text{if } \alpha + 1 = 2\beta. 
\end{cases}$$

Note that

$$\sum_{k=1}^{n} \frac{k^\alpha}{(n^2 + kn + a)^\beta} = n^{\alpha+1-2\beta} \left[ \frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{(1 + \frac{k}{n} + \frac{a}{n^2})^\beta} \right].$$

For positive $a$ and $\epsilon$ and sufficiently large $n$,

$$\frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{(1 + \frac{k}{n} + \epsilon)^\beta} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{(1 + \frac{k}{n} + \frac{a}{n^2})^\beta} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{(1 + \frac{k}{n})^\beta}.$$

Considering Riemann sums, we have, for all $\epsilon > 0$, that

$$\int_0^1 \frac{x^\alpha}{(1 + x + \epsilon)^\beta} \, dx \leq \liminf \frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{(1 + \frac{k}{n} + \frac{a}{n^2})^\beta} \leq \limsup \frac{1}{n} \sum_{k=1}^{n} \frac{(\frac{k}{n})^\alpha}{(1 + \frac{k}{n})^\beta} \leq \int_0^1 \frac{x^\alpha}{(1 + x)^\beta} \, dx.$$

Therefore

$$L(\alpha, \beta) = \lim_{n \to \infty} (n^{\alpha+1-2\beta}) \int_0^1 \frac{x^\alpha}{(1 + x)^\beta} \, dx$$

and the result follows.