PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please email your submissions to `crux-psol@cms.math.ca` *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_ProblemNumber *(example Doe_Jane_1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

Submissions of proposals. *Original problems are particularly sought, but other interesting problems are also accepted provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by someone else without permission. Solutions, if known, should be sent with proposals. If a solution is not known, some reason for the existence of a solution should be included by the proposer. Proposal files should be named using the convention* LastName_FirstName_Proposal_Year_number *(example Doe_Jane_Proposal_2014_4.tex, if this was Jane's fourth proposal submitted in 2014).*

To facilitate their consideration, solutions should be received by the editor by **1 March 2015***, although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

An asterisk ($\star$) after a number indicates that a problem was proposed without a solution.

3872. *Proposed by F. R. Ataev. Correction.*

Let x, y, z be the distances from the vertices of a triangle to its incircle and let r be the inradius of the triangle. Show that the area of the triangle is given by

$$A = \frac{\sqrt{xyz(x+2r)(y+2r)(z+2r)}}{r}\,.$$

3881. *Proposed by Ovidiu Furdui.*

Calculate

$$\sum_{n=2}^{\infty}\left(n^2\ln\left(1-\frac{1}{n^2}\right)+1\right).$$

3882. *Originally proposed by Mehmet Sahin; corrected version by Arkady Alt.*

Let ABC be a right angle triangle with $\angle CAB = 90^\circ$ and hypotenuse a. Let $[AD]$ be an altitude and let I_1 and I_2 be the incenters of the triangles ABD and ADC, respectively. Let ρ be the radius of the circle through the points B, I_1 and I_2 and let r be the inradius of the triangle ABC. Prove that

$$\rho = \sqrt{\frac{a^2 + 2ar + 2r^2}{2}}$$

and $\min \frac{\rho}{r} = \sqrt{3} + \sqrt{6}$.

3883. *Proposed by Max A. Alekseyev.*

Let a, b, c, d be positive integers such that $a + b$ and $ad + bc$ are odd. Prove that if $2^a - 3^b > 1$, then $2^a - 3^b$ does not divide $2^c + 3^d$.

3884. *Proposed by Mihai Bogdan.*

Let a, b, c and d be positive real numbers such that $a + b + c + d = k$, where $k \in (0, 8)$. Prove that:

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{d^2+1} + \frac{d}{a^2+1} \geq \frac{k(8-k)}{8}.$$

When does the equality hold?

3885. *Proposed by Oai Thanh Dao.*

Let ABC be a triangle and let F be a point that lies on a circumcircle of ABC. Further, let H_a, H_b and H_c denote projections of the orthocenter H onto sides BC, AC and AB, respectively. The three circles AH_aF, BH_bF and CH_cF meet the three sides BC, AC and AB at points A_1, B_1 and C_1, respectively. Prove that the points A_1, B_1 and C_1 are collinear.

3886. *Proposed by Michel Bataille.*

Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the nth harmonic number and let $H_0 = 0$. Prove that for $n \geq 1$, we have

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} 2^k H_k = 2H_n - H_{\lfloor n/2 \rfloor}.$$

3887. *Proposed by Dao Hoang Viet.*

Let a, b and c be positive real numbers. Prove that

$$\frac{a^2}{bc(a^2+ab+b^2)} + \frac{b^2}{ac(b^2+bc+c^2)} + \frac{c^2}{ab(a^2+ac+c^2)} \geq \frac{9}{(a+b+c)^2}.$$

3888. *Proposed by Peter Woo.*

My greatly admired high school teacher taught me one foolproof method when solving triangles. Suppose in triangle ABC you are given the measure of $\angle A$ and the lengths of the adjacent sides b and c; then to find the remaining angles in terms of the given quantities, one should use the law of cosines to find the length of the third side and then the law of sines to find measures of $\angle B$ and $\angle C$. Or so I was taught. But after many years, I found a way to solve this problems while avoiding the cosine law and the use of square roots. Can you discover such a way?

3889. *Proposed by Cristinel Mortici.*

Prove that

$$e^{\pi} > \left(\frac{e^2+\pi^2}{2e}\right)^e .$$

3890*. *Proposed by Šefket Arslanagić.*

Let $\alpha, \beta, \gamma \in \mathbb{R}$. Prove or disprove that

$$|\sin\alpha| + |\sin\beta| + |\sin\gamma| + |\cos(\alpha+\beta+\gamma)| \le 1 + \frac{3\sqrt{3}}{2}.$$

. . . .

3872. *Proposé par F. R. Ataev. Correction.*

Soit x, y et z les distances des sommets dun triangle jusquau cercle inscrit dans le triangle et soit r le rayon de ce cercle. Montrer que l'aire du triangle est donnée par

$$A = \frac{\sqrt{xyz(x+2r)(y+2r)(z+2r)}}{r} .$$

3881. *Proposé par Ovidiu Furdui.*

Calculer

$$\sum_{n=2}^{\infty}\left(n^2\ln\left(1-\frac{1}{n^2}\right)+1\right).$$

3882. *Proposé par Mehmet Sahin ; correction par Arkady Alt.*

Soit ABC un triangle rectangle en A avec hypoténuse a et soit $[AD]$ une hauteur du triangle. I_1 et I_2 sont les centres des cercles inscrits dans les triangles respectifs ABD et ADC. Soit ρ le rayon du cercle qui passe aux points B, I_1 et I_2 et soit r le rayon du cercle inscrit dans le triangle ABC. Démontrer que

$$\rho = \sqrt{\frac{a^2+2ar+2r^2}{2}}$$

et $\min \dfrac{\rho}{r} = \sqrt{3} + \sqrt{6}$.

3883. *Proposé par Max A. Alekseyev.*

Soit a, b, c, d des entiers supérieurs à 0 tels que $a + b$ et $ad + bc$ soient impairs. Démontrer que si $2^a - 3^b > 1$, alors $2^a - 3^b$ n'est pas un diviseur de $2^c + 3^d$.

3884. *Proposé par Mihai Bogdan.*

Soit a, b, c et d des réels strictement positifs tels que $a + b + c + d = k$, $k \in (0, 8)$. Démontrer que

$$\frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{d^2+1} + \frac{d}{a^2+1} \geq \frac{k(8-k)}{8}.$$

Quand y a-t-il égalité ?

3885. *Proposé par Oai Thanh Dao.*

Soit ABC un triangle et F un point sur le cercle circonscrit au triangle ABC. Soit H_a, H_b et H_c les projections de l'orthocentre H sur les côtés respectifs BC, AC et AB. Les trois cercles AH_aF, BH_bF et CH_cF coupent les côtés BC, AC et AB aux points respectifs A_1, B_1 et C_1. Démontrer que les points A_1, B_1 et C_1 sont alignés.

3886. *Proposé par Michel Bataille.*

Soit $H_n = \sum_{k=1}^{n} \frac{1}{k}$ le n^{ieme} nombre harmonique et soit $H_0 = 0$. Démontrer que pour tout n, $n \geq 1$, on a

$$\sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} 2^k H_k = 2H_n - H_{\lfloor n/2 \rfloor}.$$

3887. *Proposé par Dao Hoang Viet.*

Soit a, b et c des réels supérieurs à 0. Démontrer que

$$\frac{a^2}{bc(a^2+ab+b^2)} + \frac{b^2}{ac(b^2+bc+c^2)} + \frac{c^2}{ab(a^2+ac+c^2)} \geq \frac{9}{(a+b+c)^2}.$$

3888. *Proposé par Peter Woo.*

À l'école, une enseignante que j'admirais beaucoup nous a montré que pour déterminer la mesure des autres angles d'un triangle, étant donné la longueur de deux côtés et la mesure de l'angle compris entre eux, on détermine d'abord la longueur du troisième côté en utilisant la loi des cosinus (théorème d'Al-Kashi), puis on détermine la mesure des autres angles en utilisant la loi des sinus. Beaucoup plus tard, j'ai découvert une façon plus facile qui évite l'utilisation de la loi des cosinus et les racines carrées. Pouvez-vous découvrir une telle méthode ?

3889. *Proposé par Cristinel Mortici.*

Démontrer que

$$e^{\pi} > \left(\frac{e^2+\pi^2}{2e}\right)^e .$$

3890*. *Proposé par Šefket Arslanagić.*

Soit $\alpha, \beta, \gamma \in \mathbb{R}$. Démontrer ou infirmer que

$$|\sin\alpha| + |\sin\beta| + |\sin\gamma| + |\cos(\alpha+\beta+\gamma)| \le 1 + \frac{3\sqrt{3}}{2}.$$