THE OLYMPIAD CORNER

No. 317

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to crux-olympiad@cms.math.ca *or mail them to the address inside the back cover. Electronic submissions are preferable.*

Submissions of solutions. *Each solution should be contained in a separate file named using the convention* LastName_FirstName_OCProblemNumber *(example Doe_Jane_OC1234.tex). It is preferred that readers submit a LaTeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.*

To facilitate their consideration, solutions should be received by the editor by **1 March 2015**, *although late solutions will also be considered until a solution is published.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

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OC151. Let ABC be a triangle. The tangent at A to the circumcircle intersects the line BC at P. Let Q and R be the symmetrical of P with respect to the lines AB and AC, respectively. Prove that $BC \perp QR$.

OC152. Find all non-constant polynomials $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with integer coefficients whose roots are exactly the numbers $a_0, a_1, \ldots, a_{n-1}$ each with multiplicity 1.

OC153. Find all non-decreasing functions from the set of real numbers to itself such that for all real numbers x, y we have

$$f(f(x^2) + y + f(y)) = x^2 + 2f(y)\,.$$

OC154. For $n \in \mathbb{Z}^+$ we denote

$$x_n := \binom{2n}{n}\,.$$

Prove there exist infinitely many finite sets A, B of positive integers, such that $A \cap B = \emptyset$, and

$$\frac{\prod_{i \in A} x_i}{\prod_{j \in B} x_j} = 2012\,.$$

OC155. There are 42 students taking part in the Team Selection Test. It is known that every student knows exactly 20 other students. Show that we can divide the students into 2 groups or 21 groups such that the number of students in each group is equal and every two students in the same group know each other.

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OC151. Soit ABC un triangle et soit P le point d'intersection de a ligne BC et de la tangente du cercle circonscrit au point A. Soit Q et R symétriques à P par rapport aux lignes AB et AC respectivement. Démontrer que $BC \perp QR$.

OC152. Déterminer tous les polynômes non constants $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ avec coefficients entiers dont les racines sont exactement les nombres $a_0, a_1, \ldots, a_{n-1}$ avec les mêmes multiplicités.

OC153. Déterminer toutes les fonctions non décroissantes des nombres réels aux nombres réels telles que pour tout x, y réels on a

$$f(f(x^2) + y + f(y)) = x^2 + 2f(y)\,.$$

OC154. Pour $n \in \mathbb{Z}^+$, dénotons

$$x_n := \binom{2n}{n}.$$

Démontrer qu'il existe un nombre infini d'ensembles finis d'entiers positifs A et B, tels que $A \cap B = \emptyset$ et

$$\frac{\prod_{i \in A} x_i}{\prod_{j \in B} x_j} = 2012\,.$$

OC155. Soit 42 étudiants, où on sait que tout étudiant connait exactement 20 autres étudiants. Démontrer qu'il est possible de répartir l'ensemble des étudiants en 2 sous-ensembles ou en 21 sous-ensembles de façon à ce que chaque sous-ensemble ait le même nombre d'étudiants et que tous les étudiants dans un sous-ensemble se connaissent.