CONTEST CORNER
SOLUTIONS

CC41. Ace runs with constant speed and Flash runs \(x\) times as fast, \(x > 1\). Flash gives Ace a head start of \(y\) metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

*Originally problem 7 of 2005 W.J. Blundon Mathematics Contest.*

*Solved by R. I. Hess; and D. Văcaru. We present the solution by Daniel Văcaru.*

Let \(v\) be Ace’s speed, then Flash has speed \(vx\). Let \(t\) be the amount of time it takes for Flash to catch Ace. When Flash catches up to Ace, Ace is \(vt + y\) metres from the start and Flash is \(xvt\) metres from the start.

Thus, \(vt + y = xvt\). Solving for \(t\), we get \(t = \frac{y}{v(x-1)}\). At that time, Flash has run \(\frac{xvy}{v(x-1)} = \frac{xy}{x-1}\) metres.

CC42. \(\triangle ABC\) has its vertices on a circle of radius \(r\). If the lengths of two of the medians of \(\triangle ABC\) are equal to \(r\), determine the side lengths of \(\triangle ABC\).

*Originally 2012 Canadian Senior Mathematics Contest, problem B3c.*

*Solved by M. Amengual Covas; Š. Arslanagić; M. Bataille; M. Coiculescu; R. Hess; and D. Văcaru. We present the solution by Miguel Amengual Covas.*

Let \(G\) be the centroid of \(\triangle ABC\) and suppose that the two equal medians are the median \(AD\) to side \(BC\) and the median to side \(CA\). Clearly, then, \(\triangle ABC\) is isosceles with \(BC = CA\). Thus the median \(CM\) to side \(AB\) lies along the perpendicular bisector of chord \(AB\) and it passes through the circumcentre \(O\) of \(\triangle ABC\). Therefore, we have

\[
AO^2 - OM^2 = AG^2 - GM^2. \tag{1}
\]

Let \(GM = x\). Since \(G\) trisects each median of \(\triangle ABC\), we have \(OM = OA - MC = r - 3x\) and \(AG = \frac{2}{3}AD = \frac{2}{3}r\). When these are substituted into (1), we get \(r^2 - (r - 3x)^2 = (\frac{2}{3}r)^2 - x^2\). Solving for \(x\), we obtain \(x = \frac{2}{27}r\) (which is not admissible) and \(x = \frac{r}{12}\). Hence,

\[
AB = 2 \cdot AM = 2 \sqrt{r^2 - \left(\frac{3r}{4}\right)^2} = \frac{r \sqrt{7}}{2}
\]

and

\[
BC = CA = \sqrt{AM^2 + MC^2} = \sqrt{\left(\frac{r \sqrt{7}}{4}\right)^2 + \left(\frac{r}{4}\right)^2} = \frac{r \sqrt{2}}{2}.
\]

*Crux Mathematicorum, Vol. 39(9), November 2013*
CC43. A circle has diameter $AB$. $P$ is a fixed point of $AB$ lying between $A$ and $B$. A point $X$, distinct from $A$ and $B$, lies on the circumference of the circle. Prove that $\tan(\angle AXP) / \tan(\angle XAP)$ is constant for all values of $X$.

Originally Question 6 of 2005 APICS Math Competition.

Solved by M. Amengual Covas; Š. Arslanagić; M. Bataille; R. I. Hess; J. G. Heuver; and T. Zvonaru. We present the solution of Michel Bataille modified by the editor.

For simplicity, let $\alpha = \angle XAP$ and $\beta = \angle AXP$. Since $AB$ is a diameter, $\angle AXB = 90^\circ$ and hence $\angle BXP = 90^\circ - \beta$. Since triangle $AXB$ is right-angled with right angle at $X$, $\angle PBX = 90^\circ - \alpha$. We now apply Law of Sines on $\triangle AXP$ and $\triangle PXB$.

On $\triangle AXP$ we have $\frac{PA}{\sin \beta} = \frac{PX}{\sin \alpha}$, so

$$\frac{\sin \beta}{\sin \alpha} = \frac{PA}{PX} \quad (1)$$

On $\triangle PXB$,

$$\frac{PB}{\sin(90^\circ - \beta)} = \frac{PX}{\sin(90^\circ - \alpha)}.$$

Since $\sin(90^\circ - \beta) = \cos \beta$ and $\sin(90^\circ - \alpha) = \cos \alpha$, this implies

$$\frac{\cos \alpha}{\cos \beta} = \frac{PX}{PB} \quad (2)$$

Equations (1) and (2) imply

$$\frac{\tan(\angle AXP)}{\tan(\angle XAP)} = \frac{\tan \beta}{\tan \alpha} = \frac{\sin \beta}{\cos \beta} \cdot \frac{\cos \alpha}{\sin \alpha} = \frac{PA \cdot PX}{PB} = \frac{PA}{PB},$$

which is constant for all values of $X$.

CC44. Let $a_0 = 1$ and for $n \geq 0$ let $a_{n+1} = a_n - \frac{1}{2}a_n^2$. Find $\lim_{n \to \infty} na_n$, if it exists.

Originally Question 6 on 2009 University of Waterloo Big E Contest.

Solved by M. Bataille; and D. Văcaru. We present Michel Bataille’s solution.

We show that $\lim_{n \to \infty} na_n = 2$.

Since $a_{n+1} - a_n = -\frac{1}{2}a_n^2 < 0$ for all $n \geq 0$, the sequence $\{a_n\}$ is decreasing. It follows that $a_n \leq a_0 = 1$ for all $n \geq 0$. From $a_{n+1} = \frac{a_n}{2}(2-a_n)$, an easy induction shows that $a_n > 0$ for all $n \geq 0$. Being decreasing and bounded below, the sequence $\{a_n\}$ is convergent.

Let $\ell = \lim_{n \to \infty} a_n$. Since $\ell$ is also the limit of $\{a_{n+1}\}$, we must have $\ell = \ell - \frac{1}{2}\ell^2$ and so $\ell = 0$. Because $\frac{a_{n+1}}{a_n} = 1 - \frac{1}{2}a_n$, we have $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1$. Now, we calculate
\[
\frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{a_n - a_{n+1}}{a_n a_{n+1}} = \frac{\frac{1}{2}a_n^2}{a_n a_{n+1}} = \frac{1}{2} \cdot \frac{a_n}{a_{n+1}}
\]

and so the sequence \( \frac{1}{a_{n+1}} - \frac{1}{a_n} \) is convergent towards \( \frac{1}{2} \). The same is true of its Cesàro mean \( \{ C_n \} \) defined by

\[
C_n = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{a_n} - \frac{1}{a_{n-1}} \right).
\]

But

\[
C_n = \frac{1}{n} \left( \frac{1}{a_n} - 1 \right) = \frac{1}{n a_n} - \frac{1}{n}
\]

and so \( \lim_{n \to \infty} n a_n = \lim_{n \to \infty} \frac{1}{C_n + \frac{1}{n}} = 2. \)

**CC45.** The *baseball sum* of two rational numbers \( \frac{a}{b} \) and \( \frac{c}{d} \) is defined to be \( \frac{a + c}{b + d} \). Starting with the rational numbers \( \frac{0}{1} \) and \( \frac{1}{1} \) as Stage 0, the baseball sum of each consecutive pair of rational numbers in a stage is inserted between the pair to arrive at the next stage. The first few stages of this process are shown below:

<table>
<thead>
<tr>
<th>STAGE 0</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>STAGE 1</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>STAGE 2</td>
<td>( \frac{0}{1} )</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>STAGE 3</td>
<td>( \frac{0}{1} )</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>

Prove that:

i) no rational number will be inserted more than once,

ii) no inserted fraction is reducible, and

iii) every rational number between 0 and 1 will be inserted in the pattern at some stage.

*Originally 2006 Canadian Open Mathematics Challenge, problem B4 b).*

*One incorrect solution was received.*