CONTEST CORNER SOLUTIONS

CC41. Ace runs with constant speed and Flash runs x times as fast, $x > 1$. Flash gives Ace a head start of y metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

Originally problem 7 of 2005 W.J. Blundon Mathematics Contest.

Solved by R. I. Hess ; and D. Văcaru. We present the solution by Daniel Văcaru.

Let v be Ace's speed, then Flash has speed vx. Let t be the amount of time it takes for Flash to catch Ace. When Flash catches up to Ace, Ace is $vt + y$ metres from the start and Flash is xvt metres from the start.

Thus, $vt + y = xvt$. Solving for t, we get $t = \frac{y}{v(x-1)}$. At that time, Flash has run $\frac{xvy}{v(x-1)} = \frac{xy}{x-1}$ metres.

CC42. $\triangle ABC$ has its vertices on a circle of radius r. If the lengths of two of the medians of $\triangle ABC$ are equal to r, determine the side lengths of $\triangle ABC$.

Originally 2012 Canadian Senior Mathematics Contest, problem B3c.

Solved by M. Amengual Covas ; Š. Arslanagić ; M. Bataille ; M. Coiculescu ; R. Hess ; and D. Văcaru. We present the solution by Miguel Amengual Covas.

Let G be the centroid of $\triangle ABC$ and suppose that the two equal medians are the median AD to side BC and the median to side CA. Clearly, then, $\triangle ABC$ is isosceles with $BC = CA$. Thus the median CM to side AB lies along the perpendicular bisector of chord AB and it passes through the circumcentre O of $\triangle ABC$. Therefore, we have

$$AO^2 - OM^2 = AG^2 - GM^2. \qquad (1)$$

Let $GM = x$. Since G trisects each median of $\triangle ABC$, we have $OM = OA - MC = r - 3x$ and $AG = \frac{2}{3}AD = \frac{2}{3}r$. When these are substituted into (1), we get $r^2 - (r - 3x)^2 = (\frac{2}{3}r)^2 - x^2$. Solving for x, we obtain $x = \frac{2}{3}r$ (which is not admissible) and $x = \frac{r}{12}$. Hence,

$$AB = 2 \cdot AM = 2\sqrt{r^2 - \left(\frac{3r}{4}\right)^2} = \frac{r\sqrt{7}}{2}$$

and

$$BC = CA = \sqrt{AM^2 + MC^2} = \sqrt{\left(\frac{r\sqrt{7}}{4}\right)^2 + \left(\frac{r}{4}\right)^2} = \frac{r\sqrt{2}}{2}.$$

CC43. A circle has diameter AB. P is a fixed point of AB lying between A and B. A point X, distinct from A and B, lies on the circumference of the circle. Prove that $\tan(\angle AXP) \div \tan(\angle XAP)$ is constant for all values of X.

Originally Question 6 of 2005 APICS Math Competition.

Solved by M. Amengual Covas; Š. Arslanagić; M. Bataille; R. I. Hess; J. G. Heuver; and T. Zvonaru. We present the solution of Michel Bataille modified by the editor.

For simplicity, let $\alpha = \angle XAP$ and $\beta = \angle AXP$. Since AB is a diameter, $\angle AXB = 90^\circ$ and hence $\angle BXP = 90^\circ - \beta$. Since triangle AXB is right-angled with right angle at X, $\angle PBX = 90^\circ - \alpha$. We now apply Law of Sines on $\triangle AXP$ and $\triangle PXB$. On $\triangle AXP$ we have $\frac{PA}{\sin\beta} = \frac{PX}{\sin\alpha}$, so

$$\frac{\sin\beta}{\sin\alpha} = \frac{PA}{PX}. \tag{1}$$

On $\triangle PXB$,

$$\frac{PB}{\sin(90^\circ - \beta)} = \frac{PX}{\sin(90^\circ - \alpha)}.$$

Since $\sin(90^\circ - \beta) = \cos\beta$ and $\sin(90^\circ - \alpha) = \cos\alpha$, this implies

$$\frac{\cos\alpha}{\cos\beta} = \frac{PX}{PB}. \tag{2}$$

Equations (1) and (2) imply

$$\frac{\tan(\angle AXP)}{\tan(\angle XAP)} = \frac{\tan\beta}{\tan\alpha} = \frac{\sin\beta}{\cos\beta} \cdot \frac{\cos\alpha}{\sin\alpha} = \frac{PA}{PX}\frac{PX}{PB} = \frac{PA}{PB},$$

which is constant for all values of X.

CC44. Let $a_0 = 1$ and for $n \geq 0$ let $a_{n+1} = a_n - \frac{1}{2}a_n^2$. Find $\lim_{n\to\infty} na_n$, if it exists.

Originally Question 6 on 2009 University of Waterloo Big E Contest.

Solved by M. Bataille; and D. Văcaru. We present Michel Bataille's solution.

We show that $\lim\limits_{n\to\infty} na_n = 2$.

Since $a_{n+1} - a_n = -\frac{1}{2}a_n^2 < 0$ for all $n \geq 0$, the sequence $\{a_n\}$ is decreasing. It follows that $a_n \leq a_0 = 1$ for all $n \geq 0$. From $a_{n+1} = \frac{a_n}{2}(2 - a_n)$, an easy induction shows that $a_n > 0$ for all $n \geq 0$. Being decreasing and bounded below, the sequence $\{a_n\}$ is convergent.

Let $\ell = \lim\limits_{n\to\infty} a_n$. Since ℓ is also the limit of $\{a_{n+1}\}$, we must have $\ell = \ell - \frac{1}{2}\ell^2$ and so $\ell = 0$. Because $\dfrac{a_{n+1}}{a_n} = 1 - \frac{1}{2}a_n$, we have $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1$. Now, we calculate

$$\frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{a_n - a_{n+1}}{a_n a_{n+1}} = \frac{\frac{1}{2}a_n^2}{a_n a_{n+1}} = \frac{1}{2} \cdot \frac{a_n}{a_{n+1}}$$

and so the sequence $\frac{1}{a_{n+1}} - \frac{1}{a_n}$ is convergent towards $\frac{1}{2}$. The same is true of its Cesàro mean $\{C_n\}$ defined by

$$C_n = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{a_n} - \frac{1}{a_{n-1}} \right).$$

But

$$C_n = \frac{1}{n} \left(\frac{1}{a_n} - 1 \right) = \frac{1}{na_n} - \frac{1}{n}$$

and so $\lim\limits_{n\to\infty} na_n = \lim\limits_{n\to\infty} \frac{1}{C_n + \frac{1}{n}} = 2.$

CC45. The *baseball sum* of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is defined to be $\frac{a+c}{b+d}$. Starting with the rational numbers $\frac{0}{1}$ and $\frac{1}{1}$ as Stage 0, the baseball sum of each consecutive pair of rational numbers in a stage is inserted between the pair to arrive at the next stage. The first few stages of this process are shown below :

STAGE 0 :	$\frac{0}{1}$								$\frac{1}{1}$
STAGE 1 :	$\frac{0}{1}$				$\frac{1}{2}$				$\frac{1}{1}$
STAGE 2 :	$\frac{0}{1}$		$\frac{1}{3}$		$\frac{1}{2}$		$\frac{2}{3}$		$\frac{1}{1}$
STAGE 3 :	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{1}$

Prove that :

i) no rational number will be inserted more than once,

ii) no inserted fraction is reducible, and

iii) every rational number between 0 and 1 will be inserted in the pattern at some stage.

Originally 2006 Canadian Open Mathematics Challenge, problem B4 b).

One incorrect solution was received.