No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


(a) Find infinitely many pairs \((a, b)\) of positive rational numbers so that 
\[ \sqrt{a} - \sqrt{b} \] 
is a root of \(x^2 + ax - b\).

(b) Find two positive integers \(a, b\) so that \(\sqrt{a} - \sqrt{b}\) is a root of \(x^2 + ax - b\).

Solved by Š. Arslanagić; A. Alt; D. Bailey, E. Campbell, and C. Diminnie; M. Bataille; B. D. Beasley; M. Amengual Covas; C. Curtis; R. Hess; D. Kouka-kis; K. E. Lewis; S. Malikić; Missouri State University Problem Solving Group; M.R. Modak; C. Sánchez-Rubio; Skidmore College Problem Group; D. Smith; T. Smith; D. R. Stone and J. Hawkins; D. Văcaru; T. Zvonaru; and the proposer. There was one incorrect solution. We present a solution adapted from that of Chip Curtis.

(a) \(\sqrt{a} - \sqrt{b}\) is a root of \(x^2 + ax - b\) if and only if 
\[ a - 2\sqrt{ab} + a\sqrt{a} - a\sqrt{b} = 0, \]
or 
\[ (\sqrt{a} + 2)\sqrt{b} = \sqrt{a}(\sqrt{a} + 1). \]

Thus, we obtain the possibilities 
\[ (a, b) = \left( c^2, \frac{c^2 (c + 1)^2}{(c + 2)^2} \right), \]

where \(c\) is any positive rational.

Since 
\[ \frac{c(c + 1)}{c + 2} = c - 1 + \frac{2}{c + 2}, \]
we see that \(a\) and \(b\) cannot both be squares of integers.

However, squaring the equation relating \(a\) and \(b\) leads to 
\[ 2\sqrt{a}(2b - a) = a^2 + a - ab - 4b. \]

If \(a\) is not the square of a rational, then 
\[ a = 2b = 4b + ab - a^2 = 2b(2 - b), \]
whereupon \((a, b) = (2, 1)\). This is the only case in which \((a, b)\) is an integer pair and in which \((a, b)\) are not both squares.

(b) \(\sqrt{2} - 1\) is a root of \(x^2 + 2x - 1\).
Editor’s comment. Sánchez-Rubio noted that, if in the solution, we allow $c$ to take a negative value and the square roots to be negative (in effect, asking that $\sqrt{b} - \sqrt{a}$ be a root), then we can get integer solutions with $c = 0, -1, -3, -4$. This leads to $(a, b) = (0, 0), (1, 0), (9, 36), (16, 36)$. Thus $-1 = 0 - 1$ is a root of $x^2 + x$; $3 = 6 - 3$ is a root of $x^2 + 9x - 36 = (x + 12)(x - 3)$ and $2 = 6 - 4$ is a root of $x^2 + 16x - 36 = (x + 18)(x - 2)$. On the other hand, Zvonaru set $b = k^2a$ and obtained the equation $(1 - k)\sqrt{a} = (2k - 1)$. This leads to the family 

$$(a, b) = \left( \left( \frac{2k - 1}{1 - k} \right)^2, \left( \frac{k(2k - 1)}{1 - k} \right)^2 \right),$$

where $k$ is a rational for which $\frac{1}{2} < k < 1$. This agrees with the family given in the solution.


Given a square $ABCD$ with side length $a$. Points $K$ and $L$ are on $BC$ and $CD$, respectively, such that the perimeter of $\triangle KCL$ is $2a$. Determine the measures of the angles of $\triangle AKL$ which minimize its area.

Solved by A. Alt; Š. Arstanagić (2 solutions); D. Bailey, E. Campbell, and C. Duminie; M. Bataille; C. Curtis; J. Hawkins and D. R. Stone; R. Hess; O. Kouba; D. Koukakis; K. E. Lewis; S. Malikić; M.R.Modak; M. Parmenter; C. Sánchez-Rubio; D. Smith; I. Uchiha; T. Zvonaru; and the proposer. We present the solution by Omran Kouba.

Since the perimeter of $\triangle KCL$ is equal to $BC + CD$, we conclude that $KL = BK + DL$. Let $M$ be a point on the line $BC$ so that $BM = DL$, and $B$ is between $K$ and $M$ as in the figure:

![Diagram of the square and the points](image)

Since $AB = AD$ and $BM = DL$, the two right triangles $\triangle ABM$ and $\triangle ADL$ are congruent. In particular, $\angle BAM = \angle DAL$, thus $\angle LAM = 90^\circ$. On the other
hand, we have $AM = AL$ and $KM = KB + DL = KL$, so, $AK$ is the perpendicular bisector of a line segment $LM$. Thus $AK$ is the angle bisector of $\angle LAM$, and $\angle LAK = 45^\circ$.

Now, let $\theta = \angle KAB$, then $AK = a / \cos(\theta)$ and $AL = a / \cos(45^\circ - \theta)$, so that

$$\text{Area}(AKL) = \frac{1}{2} AK \cdot AL \cdot \sin(45^\circ) = \frac{a^2}{\sqrt{2}} \frac{1}{2 \cos(45^\circ - \theta) \cos \theta}$$

Thus, the area of $\triangle AKL$ is minimum, if and only if $\cos(45^\circ - 2\theta) = 1$, that is $\theta = 22.5^\circ$, in this case $AK = AL$ and consequently $\angle LKA = \angle KLA = 67.5^\circ$. This determines the measures of the angles of $\triangle AKL$ with minimum area, and the corresponding area is $(\sqrt{2} - 1)a^2$.


Let $R$ and $r$ be the circumradius and the inradius of a triangle with sides $a$, $b$, $c$. Under which condition on the angles of the triangle does the inequality

$$a + b + c \leq 2\sqrt{3}(R + r)$$

hold?

Solved by A. Alt; S. Brown; C. Curtis; M. Dincă; J. G. Heuver; V. Konečný; O. Kouba; K. Lau; S. Malikić; T. Zvonaru; and the proposer. We present the solution by Titu Zvonaru.

By the sine law (namely, $\sum_{\text{cyclic}} a = 2R \sum_{\text{cyclic}} \sin A$) and the formula

$$1 + \frac{r}{R} = \sum_{\text{cyclic}} \cos A,$$

the given inequality is equivalent to

$$\sum_{\text{cyclic}} \sin A \leq \sqrt{3} \sum_{\text{cyclic}} \cos A,$$

and this is exactly the inequality (1) from D.S. Mitrinović et al., Recent Advances in Geometric Inequalities, Kluwer Academic Publishers, 1989, page 256. Assuming that $A \leq B \leq C$, we have:

$$\sqrt{3} \sum_{\text{cyclic}} \cos A \leq \sum_{\text{cyclic}} \sin A \quad \text{if} \quad B \geq \frac{\pi}{3} \quad \text{(Triangle of Bager’s Type I)},$$

$$\sqrt{3} \sum_{\text{cyclic}} \cos A \geq \sum_{\text{cyclic}} \sin A \quad \text{if} \quad B \leq \frac{\pi}{3} \quad \text{(Triangle of Bager’s Type II)}.$$
Equality holds exactly when \( B = \frac{\pi}{3} \). Here is a proof. Using the formulas

\[
\sum_{\text{cyclic}} \cos^2 A = 1 - 2 \cos A \cos B \cos C
\]

and

\[
\sum_{\text{cyclic}} (\cos A \cos B - \sin A \sin B) = \sum_{\text{cyclic}} \cos(A + B) = -\sum_{\text{cyclic}} \cos A,
\]

we get:

\[
3 \left( \sum_{\text{cyclic}} \cos A \right)^2 - \left( \sum_{\text{cyclic}} \sin A \right)^2 = 3 \sum_{\text{cyclic}} \cos^2 A + 6 \sum_{\text{cyclic}} \cos A \cos B - \sum_{\text{cyclic}} \sin^2 A - 2 \sum_{\text{cyclic}} \sin A \sin B
\]

\[
= 3 \sum_{\text{cyclic}} \cos^2 A - \sum_{\text{cyclic}} (1 - \cos^2 A) + 6 \sum_{\text{cyclic}} \cos A \cos B - 2 \sum_{\text{cyclic}} \sin A \sin B
\]

\[
= 4 \sum_{\text{cyclic}} \cos^2 A - 3 + 4 \sum_{\text{cyclic}} \cos A \cos B + 2 \left( \sum_{\text{cyclic}} \cos A \cos B - \sum_{\text{cyclic}} \sin A \sin B \right)
\]

\[
= 4(1 - 2 \cos A \cos B \cos C) - 3 + 4 \sum_{\text{cyclic}} \cos A \cos B - 2 \sum_{\text{cyclic}} \cos A
\]

\[
= (1 - 2 \cos A)(1 - 2 \cos B)(1 - 2 \cos C).
\]

Since \( A \leq \frac{\pi}{3} \) and \( C \geq \frac{\pi}{3} \), it follows that \( \cos A \geq \frac{1}{2} \) and \( \cos C \leq \frac{1}{2} \), hence

\[
\sqrt{3} \sum_{\text{cyclic}} \cos A - \sum_{\text{cyclic}} \sin A
\]

has the same sign as \( 2 \cos B - 1 \), and we are done.

**3774. [2012 : 334, 336] Proposed by P. H. O. Pantoja.**

Let \( a, b, c \) be positive real numbers. Prove that

\[
\frac{c(a^2 + b^2)}{\sqrt{a^3 + b^3}} + \frac{a(b^2 + c^2)}{\sqrt{b^3 + c^4}} + \frac{b(c^2 + a^2)}{\sqrt{c^3 + a^4}} \leq \frac{3}{4}(a^2 + b^2 + c^2 + a + b + c).
\]

_Solved by A. Alt; G. Apostolopoulos; C. Curtis; O. Geupel; O. Kouba; D. Koukakis; S. Malick; P. Perfetti; D. Smith; T. Zvonaru; and the proposer. We present the solution by Arkady Alt._

We will prove the following stronger inequality:

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\[ \sum_{\text{cyclic}} \frac{c(a^2 + b^2)}{\sqrt{a^3 + b^3}} \leq \frac{\sqrt{2}}{2} (a^2 + b^2 + c^2 + a + b + c). \quad (1) \]

Note first that since \((a + b)(a^3 + b^3) - (a^2 + b^2)^2 = ab(a - b)^2 \geq 0\), we have
\[ \frac{a^2 + b^2}{\sqrt{a^3 + b^3}} \leq \sqrt{\frac{a}{a + b}}. \]

Hence
\[ \sum_{\text{cyclic}} \frac{c(a^2 + b^2)}{\sqrt{a^3 + b^3}} \leq \sum_{\text{cyclic}} c \sqrt{\frac{a}{a + b}}. \quad (2) \]

By the AM-GM Inequality, we have
\[ c^2 + \frac{a+b}{2} \geq 2 \sqrt{c^2 \left( \frac{a+b}{2} \right)} = \sqrt{2} c \sqrt{a + b}, \]
so
\[ a^2 + b^2 + c^2 + a + b + c = \sum_{\text{cyclic}} \left( c^2 + \frac{a + b}{2} \right) \geq \sqrt{2} \sum_{\text{cyclic}} c \sqrt{a + b} \quad (3) \]
or, equivalently,
\[ \sum_{\text{cyclic}} c \sqrt{a + b} \leq \frac{1}{\sqrt{2}} (a^2 + b^2 + c^2 + a + b + c). \quad (4) \]

From (2) and (4), our claim (1) follows immediately. It is easy to see that equality holds if and only if \(a = b = c = 1\).

Editor’s comment. The stronger inequality featured above was also obtained by both Malikić and Perfetti. In addition, the following stronger inequality was obtained by Geupel:
\[ \sum_{\text{cyclic}} \frac{c(a^2 + b^2)}{\sqrt{a^3 + b^3}} \leq \frac{2}{3} (a^2 + b^2 + c^2) + \frac{3}{4} (a + b + c). \]


Let \(ABCD\) be a quadrilateral with \(AC \perp BD\). Show that \(ABCD\) is cyclic if and only if \(BC \cdot AD = IA \cdot IB + IC \cdot ID\), where \(I\) is the point of intersection of the diagonals.

Solved by A. Alt; G. Apostolopoulos; Š. Arslanagić (2 solutions); M. Bataille; O. Koubâ; D. Koukakis; S. Malikić; M.R. Modak; C. Sánchez-Rubio; Skidmore College Problem Group; D. Smith; I. Uchiha; D. Văcăru; P. Y. Woo; T. Zvonaru; and the proposer. We present 2 solutions.
Solution 1 by Omran Kouba.

Let $\theta = \angle ADB$ and $\varphi = \angle ACB$. Then

\[
\frac{IA \cdot IB + IC \cdot ID}{BC \cdot AD} = \frac{ID \cdot IC + IA \cdot IB}{AD \cdot BC} = \cos \theta \cos \varphi + \sin \theta \sin \varphi
\]

Thus, the condition $BC \cdot AD = IA \cdot IB + IC \cdot ID$, is equivalent to $\cos(\theta - \varphi) = 1$ or $\theta = \varphi$, and this, in turn, is equivalent to the fact that $ABCD$ is cyclic.

Solution 2 by Itachi Uchiha.

Since $AC \perp BD$, we have $BC^2 = IB^2 + IC^2$, $AD^2 = IA^2 + ID^2$, and also $\angle IAB = \angle IDC$ if and only if $\triangle IAB \sim \triangle IDC$. Therefore

$ABCD$ is cyclic $\iff \triangle IAB \sim \triangle IDC$

$\iff IB \cdot ID = IC \cdot IA$

$\iff (IB \cdot ID - IC \cdot IA)^2 = 0$

$\iff IB^2 \cdot ID^2 + IC^2 \cdot IA^2 = 2IA \cdot IB \cdot IC \cdot ID$

$\iff IB^2 \cdot ID^2 + IC^2 \cdot IA^2 = (IA \cdot IB + IC \cdot ID)^2 - (IA^2 \cdot IB^2 + IC^2 \cdot ID^2)$

$\iff (IB^2 + IC^2)(IA^2 + ID^2) = (IA \cdot IB + IC \cdot ID)^2$

$\iff BC \cdot AD = IA \cdot IB + IC \cdot ID$,

which completes the proof.


In $\triangle ABC$ prove that

\[
\tan \left( \frac{A}{2} \right) + \tan \left( \frac{B}{2} \right) + \tan \left( \frac{C}{2} \right) \geq \frac{1}{2} \left( \sec \left( \frac{A}{2} \right) + \sec \left( \frac{B}{2} \right) + \sec \left( \frac{C}{2} \right) \right).
\]

Crux Mathematicorum, Vol. 39(8), October 2013
Solved by A. Alt; Š. Arslanagić; D. Bailey, E. Campbell, and C. Diminnie; M. Bataille; S. H. Brown; M. Dinca; O. Geupel; O. Kouba; K. Lau; S. Malikić; C. M. Quang; D. Smith; P. Y. Woo; T. Zvonaru; and the proposer. We present 5 different solutions.

Foreword. In the following solutions, for convenience let $D = A/2$, $E = B/2$ and $F = C/2$. Then $D + E + F = 90^\circ$ and $\tan D \tan E + \tan E \tan F + \tan F \tan D = 1$.

Solution 1 by Titu Zvonaru.

Since, for example, 
\[
\tan D + \tan E = \frac{\sin D \cos E + \sin E \cos D}{\cos D \cos E} = \frac{\sin(D + E)}{\cos D \cos E} = \frac{\cos F}{\cos D \cos E},
\]

the desired inequality can be rewritten as
\[
\frac{\cos D}{\cos E \cos F} + \frac{\cos E}{\cos F \cos D} + \frac{\cos F}{\cos D \cos E} \geq \frac{1}{\cos D} + \frac{1}{\cos E} + \frac{1}{\cos F}.
\]

Multiplying by $\cos D \cos E \cos F$, we see that this is equivalent to
\[
\cos^2 D + \cos^2 E + \cos^2 F \geq \cos E \cos F + \cos F \cos D + \cos D \cos E.
\]

However, this holds since
\[
(cos D - \cos E)^2 + (\cos E - \cos F)^2 + (\cos F - \cos D)^2 \geq 0.
\]

Solution 2 by Dionne Bailey, Elsie Campbell and Charles Diminnie.

For $0 < x < \frac{\pi}{2}$, let
\[
f(x) = \tan x - \frac{1}{2} \sec x - \left(x - \frac{\pi}{6}\right).
\]

Since
\[
f'(x) = \sec^2 x - \frac{1}{2} \sec x \tan x - 1 = \frac{\sin x(2\sin x - 1)}{2\cos^2 x},
\]

it follows that $f(x)$ decreases on $(0, \pi/6)$ and increases on $(\pi/6, \pi/2)$. Therefore $f(x) \geq f(\pi/6)$, so that
\[
\tan x - \frac{1}{2} \sec x \geq x - \frac{\pi}{6}
\]

for $0 < x < \pi/2$ with equality if and only if $x = \pi/6$. Since $D$, $E$ and $F$ all lie between 0 and $\pi/2$,
\[
\tan D + \tan E + \tan F - \frac{1}{2} (\sec D + \sec E + \sec F)
\]
\[
\geq \left(D - \frac{\pi}{6}\right) + \left(E - \frac{\pi}{6}\right) + \left(F - \frac{\pi}{6}\right)
\]
\[
= D + E + F - \frac{\pi}{2} = 0.
\]
Solution 3 by Cao Minh Quang.

Let \( x = \tan D \), \( y = \tan E \) and \( z = \tan F \). Taking note of the arithmetic-geometric means inequality and the fact that \( xy + yz + zx = 1 \), we find that

\[
\frac{1}{2} (\sqrt{x^2 + 1} + \sqrt{y^2 + 1} + \sqrt{z^2 + 1}) = \frac{1}{2} (\sqrt{(x + y)(x + z)} + \sqrt{(y + z)(y + x)} + \sqrt{(z + x)(z + y)}) \\
\leq \frac{1}{4} [(2x + y + z) + (2y + z + x) + (2z + x + y)] = x + y + z.
\]

This is the desired inequality.

Solution 4 by Digby Smith.

Since \( \tan^2 x \) is convex on \( (0, \pi/2) \), by Jensen’s equality, we have that

\[
\tan^2 D + \tan^2 E + \tan^2 F \geq 3 \tan^2 \frac{D + E + F}{3} = 3 \tan^2 \frac{\pi}{3} = 1.
\]

Now,

\[
[2(\tan D + \tan E + \tan F)]^2 = 4(\tan^2 D + \tan^2 E + \tan^2 F) + 8(\tan D \tan E + \tan E \tan F + \tan F \tan D) \\
\geq 1 + 3(\tan^2 D + \tan^2 E + \tan^2 F) + 8 = 3(3 + \tan^2 D + \tan^2 E + \tan^2 F) \\
= 3(\sec^2 D + \sec^2 E + \sec^2 F) \geq (\sec D + \sec E + \sec F)^2.
\]

The final step exploits Cauchy’s inequality. Thus

\[
2(\tan D + \tan E + \tan F) \geq (\sec D + \sec E + \sec F)
\]

and the desired inequality is established.

Solution 5 by Wei-Dong Jiang, the proposer.

We have:

\[
1 = \tan D \tan E + \tan E \tan F + \tan F \tan D = \tan E \tan F + \tan D (\tan E + \tan F) \\
\leq \frac{1}{4} (\tan E + \tan F)^2 + \tan D (\tan E + \tan F) \\
= \frac{1}{4} (\tan E + \tan F + 2 \tan D)^2 - \tan^2 D.
\]

Thus

\[
\tan E + \tan F + 2 \tan D \geq 2 \sqrt{1 + \tan^2 D} = 2 \sec D.
\]

Adding this to its two cyclic analogues yields the desired inequality.

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Editor’s comment. As is often the case with a trigonometry problem, there were several different approaches exhibiting a variety of efficiency and recourse to other results. Kouba produced a solution similar to the second one above and Woo analyzed the graph of \( y = 2 \tan x - \sec x \) to show that it lay above the line \( y = -\pi/3 + 2x \). Some solvers used the representation of the tangents and cosines of the half angles of a triangle in terms of the sides, semi-perimeter, inradius and area. Brown noted that the given inequality is equivalent to \( s^2 \geq 12Rr + 3r^2 \), while Lau reduced it to \( 3s^2 \leq (4R + r)^2 \). Dincă proved this generalization: Let \( A_1A_2 \ldots A_n \) be a convex \( n \)-gon. Then

\[
\sum_{k=1}^{n} \tan \frac{A_k}{2} \geq \cos \frac{\pi}{n} \sum_{k=1}^{n} \sec \frac{A_k}{2}.
\]


Let \( x, y, \) and \( z \) be positive real numbers such that \( xyz = 1 \) and \( \frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} = 3 \). Determine all possible values of \( x^4 + y^4 + z^4 \).

Solved by A. Alt; Š. Arslanagic; D. Bailey, E. Campbell and C. Diminnie; M. Bataille; C. Curtis; R. Hess; O. Kouba; D. Koukakis; S. Malikić (2 solutions); P. Perfetti; A. Plaza; C. M. Quang; D. Smith; D. R. Stone and J. Hawkins; I. Uchiha; D. Văcaru; T. Zvonaru; and the proposer. There was also an incorrect solution. We give a solution that is a composite of virtually all solutions received. By the AM-GM Inequality, we have

\[
3 = \frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4} \geq 3\sqrt[3]{\frac{1}{x^4} \cdot \frac{1}{y^4} \cdot \frac{1}{z^4}} = 3.
\]

Thus, we must have the equality above, which implies that \( \frac{1}{x^4} = \frac{1}{y^4} = \frac{1}{z^4} \) or \( x = y = z \). Since we know that \( xyz = 1 \), it follows that \( x = y = z = 1 \) and so \( x^4 + y^4 + z^4 = 3 \).


Let \( \triangle A_1A_2A_3 \) be a triangle with circumcentre \( O \), incircle \( \gamma \), incentre \( I \), and inradius \( r \). For \( i = 1, 2, 3 \), let \( A'_i \) on side \( A_iA_{i+1} \) and \( A''_i \) on side \( A_iA_{i+2} \) be such that \( A'_iA'_i \perp OA_i \) and \( \gamma \) is the \( A_i \)-excircle of \( \triangle A'_iA''_i \) where \( A_4 = A_1, A_5 = A_2 \). Prove that

(a) \( A'_1A''_1 \cdot A'_2A''_2 \cdot A'_3A''_3 = \frac{4a_1a_2a_3}{(a_1 + a_2 + a_3)^2} \cdot r^2 \)

(b) \( A'_1A''_1 + A'_2A''_2 + A'_3A''_3 = \frac{a_1^2 + a_2^2 + a_3^2}{a_1a_2a_3} \cdot IK^2 + \frac{3a_1a_2a_3}{a_1^2 + a_2^2 + a_3^2} \)

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where $a_1$, $a_2$, $a_3$ are the side lengths of $\triangle A_1A_2A_3$ and $K$ is its symmedian point.

Solved by G. Apostolopoulos; O. Geupel; O. Kouba; T. Zvonaru; and the proposer.

We present the solution by Omran Kouba.

Note that, for $i = 1, 2, 3$, we have

$$\angle A_i A_i' A_i'' = 90^\circ - \angle OA_i A_{i+1}$$

$$= 90^\circ - \frac{1}{2}(180^\circ - \angle A_i OA_{i+1})$$

$$= \frac{1}{2} \angle A_i OA_{i+1} = \angle A_i A_{i+1} A_{i+2}$$

This proves that $\triangle A_i A_i' A_i''$ and $\triangle A_i A_{i+1} A_{i+2}$ are similar. Let $h_i$ be the similarity that shrinks $\triangle A_1 A_2 A_3$ to $\triangle A_i A_i' A_i''$. If $r_i$, of radius $r_i$, is the $A_i$-excircle of $\triangle A_1 A_2 A_3$, then $h_i r_i = \gamma$ and $h_i A_{i+1} A_{i+2} = A_i' A_i''$. Thus

$$\frac{A_i' A_i''}{A_{i+1} A_{i+2}} = \frac{r}{r_i}$$

But if $s = \frac{a_1 + a_2 + a_3}{2}$ is the semi-perimeter of $\triangle A_1 A_2 A_3$, then $r_i (s - a_i) = \text{Area}(A_1 A_2 A_3) = rs$, so

$$A_i' A_i'' = a_i \left( \frac{s - a_i}{s} \right), \quad \text{for } i = 1, 2, 3. \quad (1)$$

To prove (a) we note, using Heron's formula, that

$$A_1' A_1'' \cdot A_2' A_2'' \cdot A_3' A_3'' = \frac{a_1 a_2 a_3}{s^3} (s - a_1)(s - a_2)(s - a_3)$$

$$= \frac{a_1 a_2 a_3}{s^2} \cdot \frac{s(s - a_1)(s - a_2)(s - a_3)}{s^2}$$

$$= \frac{a_1 a_2 a_3}{s^2} \cdot \frac{(sr)^2}{s^2} = \frac{a_1 a_2 a_3}{s^2} \cdot r^2,$$

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which is equivalent to (a).

To prove (b) we note first that (1) implies that

\[ A'_1 A''_1 + A'_2 A''_2 + A'_3 A''_3 = 2s - \frac{a_1^2 + a_2^2 + a_3^2}{s}. \]  

(2)

Now, we need to express \( IK^2 \) in terms of the side lengths of \( \triangle A_1 A_2 A_3 \). Since \( I \) is the barycenter of the weighted points \( (A_i; a_i) \), \( i = 1, 2, 3 \), and \( K \) is the barycenter of the weighted points \( (A_i; a_i^2) \), \( i = 1, 2, 3 \), we conclude that

\[ \overrightarrow{IK} = \sum_{i=1}^{3} \left( \frac{a_i^2}{q} - \frac{a_i}{2s} \right) \overrightarrow{OA}_i, \]

where \( q = a_1^2 + a_2^2 + a_3^2 \). But, if the circumradius of \( \triangle A_1 A_2 A_3 \) is denoted by \( R \), then \( OA_i = R \) and

\[ \overrightarrow{OA}_i \cdot \overrightarrow{OA}_{i+1} = R^2 \cos(2A_{i+2}) = R^2 - \frac{1}{2}(2R \sin A_{i+2})^2 = R^2 - \frac{a_{i+2}^2}{2}. \]

Thus,

\[ IK^2 = R^2 \left( \sum_{i=1}^{3} \left( \frac{a_i^2}{q} - \frac{a_i}{2s} \right) \right)^2 - \sum_{i=1}^{3} a_{i+2} \left( \frac{a_i^2}{q} - \frac{a_i}{2s} \right) \left( \frac{a_{i+1}^2}{q} - \frac{a_{i+1}}{2s} \right). \]

Because \( \sum_{i=1}^{3} \left( \frac{a_i^2}{q} - \frac{a_i}{2s} \right) = 0 \), the previous equation reduces to

\[ IK^2 = -a_1 a_2 a_3 \sum_{i=1}^{3} a_{i+2} \left( \frac{a_i}{q} - \frac{1}{2s} \right) \left( \frac{a_{i+1}}{q} - \frac{1}{2s} \right) \]

\[ = -a_1 a_2 a_3 \sum_{i=1}^{3} a_{i+2} \left( \frac{4a_i a_{i+1}}{q^2} + \frac{a_{i+2}}{2qs} - \frac{1}{q} + \frac{1}{4s^2} \right) \]

\[ = -a_1 a_2 a_3 \left( \frac{3a_1 a_2 a_3}{q^2} + \frac{1}{2s} - \frac{2s}{q} + \frac{2s}{4s^2} \right) \]

\[ = a_1 a_2 a_3 \left( \frac{2s}{q} - \frac{3a_1 a_2 a_3}{q^2} - \frac{1}{s} \right). \]

It follows that

\[ \frac{q}{a_1 a_2 a_3} \cdot IK^2 + \frac{3a_1 a_2 a_3}{q} = 2s - \frac{q}{s}, \]

and (b) follows from (2).

Editor’s comment. Zvonaru reported that at the MathWorld site (under the heading symmedian point) there is an expression for \( IK^2 \) that reduces to the equation in the featured solution above.
Let $\Delta ABC$ have semi-perimeter $s$ and let $x$, $y$, $z$ be the distances from the centroid to the sides $BC$, $CA$, $AB$, respectively. Prove or disprove that

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \leq s \sqrt{3}.$$ 

Solved by A. Alt; Š. Arslanagić; M. Bataille; C. Curtis; M. Dincă; O. Geupel; J. Heuver; V. Konečný; O. Kouba; S. Malikić; C. R. Pranesachar; C. Sánchez-Rubio; E. Suppa; I. Uchiha; T. Zvonaru. One incorrect solution was received. We present 2 solutions.

**Foreword.** In the following solutions, let $a$, $b$, $c$ be the sides, and $K$, $R$, $r$ the area, circumradius, and inradius of the triangle. Let $h_a$, $h_b$, and $h_c$ be the altitudes of the triangle.

**Solution 1 by Itachi Uchiha.**

Since $\frac{1}{3}K = \frac{1}{2}ah_a$, we have $x = \frac{2K}{3a}$. Similarly, $y = \frac{2K}{3b}$ and $z = \frac{2K}{3c}$. By the RMS-AM inequality (or Power Mean inequality), the formula $K = rs = \frac{abc}{4R}$, and Euler’s inequality $R > 2r$, we have

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = \frac{2K}{3} \left( \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \right)$$

$$\leq \frac{2K}{3} \sqrt{\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}} = \frac{2K}{3} \sqrt{\frac{a+b+c}{abc}}$$

$$= \frac{2rs}{3} \sqrt{\frac{2s}{4Rrs}} = \frac{s}{\sqrt{3}} \sqrt{\frac{2s}{R}} \leq \frac{s}{\sqrt{3}}.$$ 

with equality if and only if the triangle is equilateral.

**Solution 2 is a composite of similar solutions by Šefket Arslanagić, Salem Malikić and Titu Zvonaru.**

It is readily verified that $x = \frac{1}{3}h_a$, $y = \frac{1}{3}h_b$, $z = \frac{1}{3}h_c$. By the AM-GM inequality,

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \leq x + y + z = \frac{1}{3} (h_a + h_b + h_c),$$

so it suffices to prove that

$$h_a + h_b + h_c \leq s\sqrt{3}.$$ 

This last inequality is item 6.1 on page 60 of O. Bottema et al, *Geometric Inequalities*, Wolters Noordhoff, 1969.

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Let \( f : [0, 1] \to \mathbb{R} \) be a continuously differentiable function and let
\[
x_n = f \left( \frac{1}{n} \right) + f \left( \frac{2}{n} \right) + \cdots + f \left( \frac{n-1}{n} \right).
\]
Calculate \( \lim_{n \to \infty} (x_{n+1} - x_n) \).

Solved by A. Alt; M. Bataille; O. Kouba; M. R. Modak; P. Perfetti; and the proposer. There were five flawed solutions, three of which applied an invalid converse of the Stolz-Cesaro theorem. We present 2 solutions.

Solution 1 by Omran Kouba.

The required limit is equal to \( \int_0^1 f(x)dx \).

We first note that, if \( g : [0, 1] \to \mathbb{R} \) is a continuously differentiable function, then, using integration by parts, we have that
\[
\int_0^1 \left( x - \frac{1}{2} \right) g'(x)dx = \left[ \left( x - \frac{1}{2} \right) g(x) \right]_0^1 - \int_0^1 g(x)dx = \frac{g(1) + g(0)}{2} - \int_0^1 g(x)dx.
\]

Apply this to the function \( g(x) = f((k + x)/n) \) for \( k = 0, 1, \ldots, n-1 \) and add the resulting equations to obtain
\[
x_n + \frac{f(0) + f(1)}{2} - n \int_0^1 f(x)dx = \int_0^1 \left( x - \frac{1}{2} \right) H_n(x)dx,
\]
where
\[
H_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f' \left( \frac{k + x}{n} \right).
\]

Observe that, for each \( x \in [0, 1] \), \( H_n(x) \) is a Riemann sum for the integral \( \int_0^1 f'(t)dt \) and \( |H_n(x)| \leq \sup_{[0,1]} |f'| \).

From the foregoing equation and its analogue for \( n + 1 \), we obtain that
\[
x_{n+1} - x_n - \int_0^1 f(x)dx = \int_0^1 (H_{n+1}(x) - H_n(x)) \left( x - \frac{1}{2} \right) dx.
\]
As \( n \) tends to infinity, the integrand on the right side tends pointwise and boundedly to 0, so by the Lebesgue Dominated Convergent Theorem, we conclude that \( \lim_{n \to \infty} (x_{n+1} - x_n) = \int_0^1 f(x)dx \).
Solution 2 by Paolo Perfetti.

There exists \( \xi_k \in (k(n + 1)^{-1}, kn^{-1}) \) for which

\[
x_{n+1} - x_n = \sum_{k=1}^{n} \left( f\left( \frac{k}{n+1} \right) - f\left( \frac{k}{n} \right) \right) + f(1)
\]

\[
= \sum_{k=1}^{n} f'(\xi_k) \left( \frac{-k}{n(n+1)} \right) + f(1)
\]

\[
= -\frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} \left( f'(\xi_k) - f'\left( \frac{k}{n} \right) \right) - \frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} f'\left( \frac{k}{n} \right) + f(1)
\]

where \( k(n + 1)^{-1} < \xi_k < kn^{-1} \). Note that \( f' \) is uniformly continuous on \([0, 1]\) and that

\[
\left| \xi_k - \frac{k}{n} \right| \leq \frac{k}{n(n+1)} < \frac{1}{n}.
\]

Therefore, for each \( \epsilon > 0 \), when \( n \) is sufficiently large

\[
\left| f'(\xi_k) - f'\left( \frac{k}{n} \right) \right| < \epsilon
\]

for \( 1 \leq k \leq n \). Thus

\[
\left| -\frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} \left( f'(\xi_k) - f'\left( \frac{k}{n} \right) \right) \right| < \epsilon \left( \frac{1}{n(n+1)} \right) \frac{n(n+1)}{2} = \frac{\epsilon}{2}.
\]

Moreover

\[
\lim_{n \to \infty} -\frac{1}{n+1} \sum_{k=1}^{n} \frac{k}{n} f'\left( \frac{k}{n} \right) = -\int_{0}^{1} xf'(x)dx.
\]

Therefore, integrating by parts, we find that

\[
\lim_{n \to \infty} (x_{n+1} - x_n) = f(1) - \int_{0}^{1} xf'(x)dx = \int_{0}^{1} f(x)dx.
\]

Editor’s comment. Malikić and Ricardo noted that the proposer poses and solves this problem in his book Limits, Series, and Fractional Part Integrals published by Springer in 2013. It is problem 1.32 on page 6 under Miscellaneous Limits; the solution appears on page 52.

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