CC90. For a given $k > 0$, $n \geq 2k > 0$, consider the square $R$ in the plane consisting of all points $(x, y)$ with $0 \leq x, y \leq n$. Color each point in $R$ gray if $\frac{x+y}{k} \leq x + y$, and blue otherwise. Find the area of the gray region in terms of $n$ and $k$.

CONTEST CORNER

SOLUTIONS

CC36. For each positive integer $n$, define $f(n)$ to be the smallest positive integer $s$ for which $1 + 2 + 3 + \cdots + (s - 1) + s$ is divisible by $n$. For example, $f(5) = 4$, because $1 + 2 + 3 + 4$ is divisible by 5 and none of $1, 1 + 2,$ or $1 + 2 + 3$ is divisible by 5. Determine, with proof, the smallest positive integer $k$ for which the equation $f(n) = f(n + k)$ has an odd positive integer solution for $n$.

Originally Question B4 c) on 2009 Canadian Open Mathematics Challenge.

One incorrect solution was received.

CC37. $ABCD$ is a cyclic quadrilateral, with side $AD = d$, where $d$ is the diameter of the circle. $AB = a, BC = a$ and $CD = b$. If $a, b$ and $d$ are integers $a \neq b$,

a) prove that $d$ cannot be a prime number.

b) determine the minimum value of $d$.

Originally Question 10 on 1999 Euclid contest.

Solved by Šefket Arslanagić and we present his solution.

Join $A$ to $C$ and since $\angle ACD$ is in a semicircle, we have $\angle ACD = 90^\circ$. Let $\angle ABC = \alpha$, so that $\angle CDA = 180^\circ - \alpha$ because $ABCD$ is a cyclic quadrilateral. From $\triangle ABC$, $AC^2 = a^2 + a^2 - 2a^2 \cos \alpha$. Similarly, from $\triangle ACD$, $AC^2 = d^2 - b^2$ and $\cos(180^\circ - \alpha) = \frac{-b}{d}$. By substitution,

$$d^2 - b^2 = 2a^2 - 2a^2 \left( \frac{-b}{d} \right).$$

Simplifying yields $2a^2 = d(d - b)$, $d \neq b$.

a) To prove $d$ cannot be prime, we consider a contradiction and suppose $d$ is prime. We have two cases from which we deduce $d$ is composite: $d = 2$ or $d \geq 3$.

If $d = 2$ then $2a^2 = 2(2 - b)$ and hence $b + a^2 = 2$. Since $a, b$ are positive integers, this implies $a = b = 1$ which is not possible since $a \neq b$. 

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Suppose \( d \geq 3 \). Since \( d > 2 \), \( d \mid a^2 \). Since \( d \) is prime, this implies \( d \mid a \), which is impossible because \( d \) is the diameter of the circle and is larger than \( a \).

b) We know \( d \geq 2 \) and can not be prime by part a), so let us check cases.

- If \( d = 4 \), we have \( a^2 = 2(4 - b) \). If \( b = 1 \) or \( 3 \), then \( a^2 = 6 \) or \( 2 \) which implies \( a \) is not an integer. If \( b = 2 \) then \( a = 2 \) but \( a \neq b \) so this is impossible. Hence \( d = 4 \) is impossible.
- If \( d = 6 \), we have \( a^2 = 3(6 - b) \), so \( b \in \{1, 2, 3, 4, 5\} \). If \( b \in \{1, 2, 4, 5\} \) then \( a \) is not an integer. If \( b = 3 \) then \( a = 3 \), but as before, \( a \neq b \). Thus \( d = 6 \) is also impossible.
- If \( d = 8 \) then \( a^2 = 4(8 - b) \). If we set \( b = 7 \), we have \( a = 2 \) and this is an acceptable solution, so the minimum possible value of \( d \) is 8.

CC38. Each vertex of a regular 11-gon is coloured black or gold. All possible triangles are formed using these vertices. Prove that there are either two congruent triangles with three black vertices or two congruent triangles with three gold vertices.

*Originally Question 5 on 2011 Sun Life Financial Repêchage Competition.*

*Solved by G. Geupel; and T. Zeornaru and N. Stanciu. We present the solution of Gesine Geupel.*

By the Pigeonhole Principle, we can find 6 vertices of the same colour. There are \( \binom{11}{2} = 55 \) lines joining pairs of these points. A regular 11-gon has 11 axes of symmetry, and each of these lines is parallel to one axis of symmetry, so by the Pigeonhole Principle some two lines are parallel. We get two congruent triangles by considering the two vertices of one line with each vertex on the other line.

CC39. Charles is playing a variant of Sudoku. To each lattice point \((x, y)\) where \(1 \leq x, y < n\), he assigns an integer between 1 and \(n\), inclusive, for some positive integer \(n\). These integers satisfy the property that in any row where \(y = k\), the \(n - 1\) values are distinct and are never equal to \(k\); similarly for any column where \(x = k\). Now, Charles randomly selects one of his lattice points with probability proportional to the integer value he assigned to it. Compute the expected value of \(x + y\) for the chosen point \((x, y)\).

*Originally Question 9 on 2013 Stanford Math Tournament, Team Problems.*

No solutions were received.

CC40. Define \( P(1) = P(2) = 1 \) and \( P(n) = P(P(n - 1)) + P(n - P(n - 1)) \) for \( n \geq 3 \). Prove that \( P(2n) \leq 2P(n) \) for all positive integers \( n \).

*Originally Question 6 on 2007 University of Waterloo Big E Contest.*

One incorrect solution was received.