PROBLEM OF THE MONTH
No. 7
Diane and Roy Dowling

This column is dedicated to the memory of former CRUX with MAYHEM Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The “Problem of the Month” features a problem and solution that we know Jim would have liked.

The Lasting Legacy of Ludolph Lehmus

This article appeared in Manitoba Math Links, Volume II, Issue #3, Spring 2002. The editor thanks the Mathematics Department at the University of Manitoba for allowing us to reproduce it.

Diane Mary Dowling (1933-2005) and Roy Dowling served as members of the Department of Mathematics at the University of Manitoba for over 40 years contributing to both research in mathematics and mathematics education as a whole. In 2006, Roy Dowling established the Diane Dowling Memorial Scholarship to honour his wife’s academic legacy in mathematics.

In the early nineteenth century an interesting problem came to the attention of those who enjoyed geometry. It has been said of this problem that its beauty lies in the simplicity of its statement and in the difficulty of its solution. Before looking at it, let us consider a problem you may be familiar with:

If in the following diagram $AB = AC$, $BD$ bisects $\angle ABC$ and $CE$ bisects $\angle ACB$, prove that $BD = CE$.

When you have solved this problem, you have proved the statement:

*If a triangle is isosceles then two of its internal bisectors are equal.*

In about 1840, a question occurred to a Berlin professor, Ludolph Lehmus: is the converse of this statement true? The converse is:

*If two internal bisectors of a triangle are equal then the triangle is isosceles.*

Copyright © Canadian Mathematical Society, 2014
People have thought about the properties of triangles for thousands of years so it is amazing that there is no record of anyone considering this converse before Professor Lehmus did. He approached Jacob Steiner with his question. This famous geometer was soon able to establish the truth of the converse above and it came to be known as the Steiner-Lehmus Theorem. Before very long Professor Lehmus himself found a nicer proof. Since that time geometry hobbyists have been fascinated by the search for simple and neat proofs of the theorem. You can see some of these proofs on the website:

http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/steiner-lehmus

In the 1960s Martin Gardner, magician and puzzle enthusiast, who regularly contributed to the Scientific American, discussed the Steiner-Lehmus Theorem in one of his columns. This column stimulated a lot of interest and hundreds of readers sent in their own proofs. Martin Gardner examined all these proofs and selected his favourite. This very nice proof was presented by two British engineers, G. Gilbert and D. MacDonnell. A few years later someone searched for the proof that Ludolph Lehmus had found over a hundred years previously and discovered that it was essentially the same as that of Gilbert and MacDonnell! If you would like to see their proof go to the website:

http://poncelet.math.nthu.edu.tw/disk5/js/geometry/geometry.html

The publication of the Steiner-Lehmus Theorem not only got people trying to find neat proofs of the theorem itself but also got them thinking about variations on its theme. For example, is there a corresponding theorem about internal trisectors? That is, if in the following diagram $\angle CBD = \frac{1}{3}\angle CBA$, $\angle BCE = \frac{1}{3}\angle BCA$ and $BD = CE$, can it be shown that $AB = AC$?

\[
\begin{array}{c}
A \\
B \\
C \\
\end{array}
\qquad
\begin{array}{c}
D \\
E \\
\end{array}
\]

The answer is yes. In fact, the $\frac{1}{3}$ may be replaced by any fraction between 0 and 1. The proof is not easy.

Another variation involves exterior angles of a triangle. Consider the following diagram:

For a triangle $ABC$, if the bisector of an exterior angle at $B$ meets the side $AC$ extended at the point $F$ then the line segment $BF$ is called the external bisector at $B$. For the triangle $ABC$ in the next diagram the bisector of the exterior angle $\angle ABU$ meets the side $AC$ extended at $F$ so $BF$ is the external bisector at $B$. The bisector of the exterior angle $\angle ACV$ meets the side $AB$ extended at $G$, so $CG$ is

Crux Mathematicorum, Vol. 39(7), September 2013
the external bisector at \( C \).

It is not hard to prove that if \( AB = AC \), then \( BF = CG \). In other words, if a triangle is isosceles then two of its external bisectors are equal. The converse of this statement is:

*If two external bisectors of a triangle are equal then the triangle is isosceles.*

At first sight this statement looks very plausible. However, it is not true. A. Emmerich pointed out the surprising fact that a triangle whose interior angles are 132°, 36° and 12° has two of its external bisectors equal. A triangle having these angles is referred to as an Emmerich triangle. To see why an Emmerich triangle has two equal external bisectors consider the following diagram.

Triangle \( ABC \) is an Emmerich triangle with \( \angle CAB = 36^\circ \), \( \angle ABC = 132^\circ \) and \( \angle BCA = 12^\circ \). The bisector of the exterior angle \( \angle ABU \) meets the side \( AC \) extended at \( F \) so \( BF \) is the external bisector at \( B \). The bisector of the exterior angle \( \angle ACV \) meets the side \( AB \) extended at \( G \) so \( CG \) is the external bisector at \( C \). We will show that the external bisector \( BF \) equals the external bisector \( CG \).

\[
\angle FBA = \frac{1}{2}(180^\circ - 132^\circ) = 24^\circ; \\
\angle FBC = \angle FBA + \angle CBA = 24^\circ + 132^\circ = 156^\circ.
\]
Now consider triangle $BCF$:
\[
\angle BCF = 12^\circ \quad \text{and} \quad \angle BFC = 180^\circ - \angle FBC - \angle BCF = 180^\circ - 156^\circ - 12^\circ = 12^\circ.
\]
Since $\angle BFC = \angle BCF$, triangle $FBC$ is isosceles with $BF = BC$. Now consider triangle $BCG$:
\[
\angle BCG = \frac{1}{2}(180^\circ - 12^\circ) = 84^\circ, \quad \angle GBC = 180^\circ - 132^\circ = 48^\circ.
\]
\[
\angle BGC = 180^\circ - \angle BCG - \angle GBC = 180^\circ - 84^\circ - 48^\circ = 48^\circ.
\]
Since $\angle GBC = \angle BGC$, triangle $BCG$ is isosceles with $CG = BC$. Since $BF$ and $CG$ both equal $BC$ these two external bisectors are equal to each other.

Over 160 years have passed since Ludolph Lehmus posed his question and popular mathematics magazines are still publishing articles with new proofs of the Steiner-Lehmus Theorem or its generalizations. For example, the October 2001 issue of *The American Mathematical Monthly* carried an article called *Other Versions of the Steiner-Lehmus Theorem*. We can certainly say that for amateur geometers the legacy of Ludolph Lehmus lives on!

Diane and Roy Dowling
Department of Mathematics
University of Manitoba
Winnipeg, MB

---

**ATOM Volume II: Algebra Intermediate Methods**
by Bruce L.R. Shawyer.

This volume contains a selection of some of the basic algebra that is useful in solving problems at the senior high school level. Many of the problems in the booklet admit several approaches. Some worked examples are shown, but most are left to the ingenuity of the reader.

There are currently 13 booklets in the series. For information on tiles in this series and how to order, visit the ATOM page on the CMS website:


---

*Crux Mathematicorum*, Vol. 39(7), September 2013