THE CONTEST CORNER

No. 17
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The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please email your submissions to crux-contest@cms.math.ca or mail them to the address inside the back cover. Electronic submissions are preferable.

Submissions of solutions. Each solution should be contained in a separate file named using the convention LastNameFirstName_CCProblemNumber (example Doe_Jane_OC1234.tex). It is preferred that readers submit a \TeX file and a pdf file for each solution, although other formats are also accepted. Submissions by regular mail are also accepted. Each solution should start on a separate page and name(s) of solver(s) with affiliation, city and country should appear at the start of each solution.

To facilitate their consideration, solutions should be received by the editor by 1 December 2014, although late solutions will also be considered until a solution is published.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions’ section, the problem will be stated in the language of the primary featured solution.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC81. Quadrilateral \(ABCD\) has the following properties:

1. the mid-point \(O\) of side \(AB\) is the centre of a semicircle;
2. sides \(AD\), \(DC\) and \(CB\) are tangent to this semicircle.

Prove that \(AB^2 = 4AD \times BC\).

CC82. For each positive integer \(N\), an Eden sequence from \(\{1, 2, 3, \ldots, N\}\) is defined to be a sequence that satisfies the following conditions:

1. each of its terms is an element of the set of consecutive integers \(\{1, 2, 3, \ldots, N\}\);
2. the sequence is increasing, and
3. the terms in odd numbered positions are odd and the terms in even numbered positions are even.

For example, the four Eden sequences from \(\{1, 2, 3\}\) are

1 3 1, 2 1, 2, 3

For each positive integer \(N\), define \(e(N)\) to be the number of Eden sequences from \(\{1, 2, 3, \ldots, N\}\). If \(e(17) = 4180\) and \(e(20) = 17710\), determine \(e(18)\) and \(e(19)\).
CC83. A map shows all Beryls Llamaburgers restaurant locations in North America. On this map, a line segment is drawn from each restaurant to the restaurant that is closest to it. Every restaurant has a unique closest neighbour. (Note that if \(A\) and \(B\) are two of the restaurants, then \(A\) may be the closest to \(B\) without \(B\) being closest to \(A\).) Prove that no restaurant can be connected to more than five other restaurants.

CC84. Let \(m\) and \(n\) be odd positive integers. Each square of an \(m\) by \(n\) board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of \(m\) and \(n\).

CC85. While Lino was simplifying the fraction \(\frac{A^3 + B^3}{A^3 + C^3}\) he cancelled the threes \(\frac{A^3 + B^3}{A^3 + C^3}\) to obtain the fraction \(\frac{A + B}{A + C}\). If \(B \neq C\), determine a necessary and sufficient condition on \(A\), \(B\) and \(C\) for Lino’s method to actually yield the correct answer, i.e. for

\[
\frac{A^3 + B^3}{A^3 + C^3} = \frac{A + B}{A + C}
\]

CC81. On considère un quadrilatère \(ABCD\) dont :

1. le milieu \(O\) du côté \(AB\) est le centre d’un demi-cercle ;
2. les côtés \(AD\), \(DC\) et \(CB\) sont tangents à ce demi-cercle.

Démontrer que \(AB^2 = 4AD \times BC\).

CC82. Étant donné un entier strictement positif \(N\), une suite Eden sur l’ensemble \(\{1, 2, 3, \ldots, N\}\) des entiers consécutifs de 1 à \(N\) est une suite qui satisfait aux conditions suivantes :

1. chacun de ses termes est un élément de l’ensemble \(\{1, 2, 3, \ldots, N\}\),
2. la suite est croissante et
3. les termes dans les positions impaires sont impairs et les termes dans les positions paires sont pairs.

Par exemple, les quatre suites Eden sur l’ensemble \(\{1, 2, 3\}\) sont :

\[
\begin{array}{cccc}
1 & 3 & 1 & 2 \\
& & 1, 2, 3
\end{array}
\]
Étant donné un entier strictement positif $N$, soit $e(N)$ le nombre de suites Eden sur l'ensemble $\{1, 2, 3, \ldots, N\}$. Sachant que $e(17) = 4180$ et $e(20) = 17710$, déterminer $e(18)$ et $e(19)$.

**CC83.** Une carte indique où sont situés tous les restaurants *La poutine dorée* en Amérique du nord. Sur cette carte, on a tracé un segment entre chaque restaurant et le restaurant qui est plus près de lui. Chaque restaurant a un seul voisin le plus près. (On remarquera qu’il est possible qu’un restaurant $A$ soit le plus près de $B$ sans que $B$ soit le restaurant le plus près de $A$.) Démontrer qu’il est impossible pour un restaurant d’être relié par des segments à plus de cinq autres restaurants.

**CC84.** Soit $m$ et $n$ deux entiers impairs positifs. Chaque case d’un quadrillage $m$ sur $n$ est colorié en rouge ou en bleu. On dit qu’une rangée du quadrillage est à dominance rouge si la rangée contient plus de cases rouges que de cases bleues. On dit qu’une colonne est à dominance bleue si la colonne contient plus de cases bleues que de cases rouges. Déterminer la valeur maximale possible de la somme de rangées à dominance rouge et de colonnes à dominance bleue. Exprimer sa réponse en fonction de $m$ et de $n$.

**CC85.** Pour simplifier l’expression $\frac{A^3 + B^3}{A^3 + C^3}$, Lino a annulé les exposants 3, en faisant $\frac{A^2 + B^2}{A^2 + C^2}$, pour obtenir l’expression $\frac{A + B}{A + C}$. Si $B \neq C$, déterminer une condition nécessaire et suffisante sur les variables $A$, $B$ et $C$ pour que la méthode de Lino soit valable, c’est-à-dire pour que

$$\frac{A^3 + B^3}{A^3 + C^3} = \frac{A + B}{A + C}.$$
CONTEST CORNER
SOLUTIONS

CC31. Triangle ABC is right angled with its right angle at A. The points P and Q are on the hypotenuse BC such that \( BP = PQ = QC \), \( AP = 3 \) and \( AQ = 4 \). Determine the length of each side of \( ABC \).

(Originally Question B3 from 1999 Canadian Open Mathematics Challenge.)

Solved by \( \acute{S}. \) Arslanagić; M. Bataille; M. Coiculescu; C. Curtis; J. G. Heuver; R. Hess; M. Stoênesescu; D. Văcaru; and T. Zvonaru. We give the solution of Chip Curtis modified by the editor.

Let \( a = BC \), \( b = CA \), \( c = AB \), \( d = BP = PQ = QC \), so \( a = 3d \). Applying the Law of Cosines on angle \( C \) in triangle \( ACQ \) and angle \( B \) on triangle \( ABP \) we have

\[
16 = b^2 + d^2 - 2bd \cdot \frac{b}{a},
\]

and

\[
9 = c^2 + d^2 - 2cd \cdot \frac{c}{a}.
\]

Substituting \( a = 3d \) in both equations above we get

\[
48 = b^2 + 3d^2
\]

and

\[
27 = c^2 + 3d^2.
\]

Furthermore, Pythagorean Theorem on \( ABC \) gives us

\[
b^2 = c^2 + 9d^2.
\]

Equations (3), (4), (5) are linear in \( b^2, c^2, d^2 \). Solving we get \( b = \sqrt{33}, c = 2\sqrt{3}, d = \sqrt{5} \) and hence \( a = 3\sqrt{5} \).

CC32. Four boys and four girls each bring one gift to a Christmas gift exchange. On a sheet of paper, each boy randomly writes down the name of one girl, and each girl randomly writes down the name of one boy. At the same time, each person passes their gift to the person whose name is written on their sheet. Determine the probability that both of these events occur:

(i) Each person receives exactly one gift;

(ii) No two people exchanged presents with each other (i.e., if A gave his gift to B, then B did not give her gift to A).

(Originally Question 4 from 2013 Sun Life Financial Repêchage Competition.)

Solved by C. Curtis; G. Geupel; and T. Zvonaru. We present Gesine Geupel’s solution below.
Let $A, B, C, D$ be the girls and $a, b, c, d$ be the boys. The different possibilities of exchanging presents are presented in the figures below. The number of choices is written beside each of the arrows.

Let us consider the girl, $A$. $A$ gives a present to a boy, $a$. There are 4 possibilities for the choice of boy $a$. $a$ gives a present to a girl, $B$, that cannot be $A$. There are 3 possible choices for $B$. $B$ gives a present to a boy, $b$, who is different from $a$ (3 possibilities). Now there are two cases.

In the first case, $b$ gives a gift to a girl, $C$, distinct from $A$ and $B$. This is shown in the left figure. Then $C$ gives a present to $c$. Now $c$ must give to $D$ and $D$ to $d$. $d$ gives to $A$. In the second case, $b$ gives a present to $A$. Now in the group, $C, c, D, d$ they do the same 4-cycle pattern; but $C$ has 2 possibilities for choosing $c$ and then the cycle is fixed.

These are the only possibilities where each person gets a present, but no two people exchange with each other. For the first picture, there are $4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 144$ possibilities; for the second, there are $4 \cdot 3 \cdot 3 \cdot 2 = 72$ possibilities; so there are 216 possibilities. The total possible cases are $8^8$ because each of the eight persons have 4 possible people of opposite gender to choose. So the probability is $\frac{216}{8^8} = \frac{27}{8192}$.

**CC33.** The abundancy index $I(n)$ of a positive integer $n$ is $I(n) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of all positive integer divisors of $n$, including 1 and $n$ itself. For example, $I(12) = \frac{1+2+3+4+6+12}{12} = \frac{7}{3}$. Determine, with justification, the smallest odd positive integer $n$ such that $I(n) > 2$.

*(Originally Question 4 from 2006 Hypatia Contest.)*

**Solution adapted from the solution of Chip Curtis.**

Let $n$ be a positive integer whose prime factorization is $p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ where $p_i$’s are primes and $e_i$’s are positive integers. Then the factors of $n$ are all of the form $p_1^{j_1} p_2^{j_2} \cdots p_m^{j_m}$ where $0 \leq j_i \leq e_i$ for every $i$. Thus, an explicit expression for $I(n)$ in terms of its prime divisors is

$$
\frac{1}{n} \sum_{j_1=0}^{e_1} \cdots \sum_{j_m=0}^{e_m} p_1^{j_1} p_2^{j_2} \cdots p_m^{j_m} = \frac{1}{n} \prod_{i=1}^{m} \left( \sum_{j_i=0}^{e_i} p_i^{j_i} \right) = \prod_{i=1}^{m} \left( \frac{\sum_{j_i=0}^{e_i} p_i^{j_i}}{p_i^{e_i}} \right) = \prod_{i=1}^{m} I(p_i^{e_i}).
$$

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We make some statements that we appeal to later:

- If \( p, q \) are primes with \( p < q \), then \( I(p^k) > I(q^k) \) for any positive integer \( k \).
  
  To see this, observe that for any prime \( r \),
  
  \[
  I(r^k) = \sum_{j=0}^{k} \frac{1}{r^j} = \sum_{j=0}^{k} \frac{1}{r^j} 
  \]

  and if \( p < q \) then \( \frac{1}{p^j} > \frac{1}{q^j} \) for any positive \( j \).

- By the multiplicativity of \( I \) and the previous comment, if \( n \) is a positive integer, we can find a positive integer \( m \) with \( I(m) > I(n) \) by replacing a prime in the prime decomposition of \( n \) with one that is smaller.

- For any prime \( p \),
  
  \[
  I(p^k) = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^k} = \frac{1 - (1/p)^{k+1}}{1 - 1/p} < \frac{1}{1 - 1/p} = \frac{p}{p-1}. 
  \]

- From the above, if \( n \) is odd and has two distinct prime factors, then maximizing \( I(n) \), we require \( n = 3^a 5^b \) for some positive integers \( a, b \). From this we have
  
  \[
  I(n) = I(3^a 5^b) = I(3^a)I(5^b) < \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8} < 2. 
  \]

- Thus, if \( n \) is an odd integer with \( I(n) > 2 \), \( n \) has at least three distinct prime factors.

- A quick check shows
  
  \[
  I(3 \cdot 5 \cdot 7) = \frac{64}{35} < 2, \quad I(3^2 \cdot 5 \cdot 7) = \frac{208}{105} < 2, \quad I(3^3 \cdot 5 \cdot 7) = \frac{128}{63} > 2, 
  \]

  so \( n = 945 \) satisfies \( I(n) > 2 \). To show 945 is the smallest, observe:

  - Any positive odd integer with at least 4 prime factors is at least \( 3 \cdot 5 \cdot 7 \cdot 11 = 1155 > 945 \).

  - From what we mentioned earlier then, the smallest \( n \) must have 3 distinct prime divisors and hence, to minimize \( n \), must have 3, 5, 7 as its distinct prime divisors.

  - We only need to consider integers \( 3^a 5^b 7^c \) with \( a \geq b \geq c \) since otherwise we could reassign the exponents and obtain a smaller integer.

  - We have checked all integers less than 945 of the form \( 3^a 5^b 7^c \) with \( a \geq b \geq c \), so we are done.

**CC34.** At the Mathville Dim Sum restaurant, all the dishes come in three sizes: small, medium, and large. Small dishes cost \( x \), medium dishes cost \( y \), and large dishes cost \( z \), where \( x, y, z \) are positive integers, with \( x < y < z \). At this
restaurant, there is no tax on any dish, and the prices haven’t changed for a long time. Margaret, Art, and Edgar had dinner there last night, and together, they ordered 9 small dishes, 6 medium dishes, and 8 large dishes. When the bill came, the following conversation ensued:

Margaret: “This bill is exactly twice as much as when I last came here.”

Art: “This bill is exactly three times as much as when I last came here.”

Edgar: “Oh, that was a delicious meal, and very reasonably priced too. Even if we give the waiter a 10% tip, the total is still less than $100.”

Determine the values of $x$, $y$, and $z$.

(Originally Question 7 from 2002 APICS Math Competition.)

Solved by C. Curtis; G. Geupel; and R. Hess. We present Chip Curtis’s solution.

We claim that the unique solution is $(x, y, z) = (2, 3, 6)$.

Letting $m$ be the amount of Margaret’s previous bill, and $a$ the amount of Art’s previous bill, we have

$$9x + 6y + 8z = 2m$$  \hspace{1cm} (1)

$$9x + 6y + 8z = 3a$$  \hspace{1cm} (2)

and

$$9x + 6y + 8z < \frac{100}{1.1} < 91$$  \hspace{1cm} (3)

By (1), $x$ is even, implying that $x \geq 2$, $y \geq 3$, and $z \geq 4$. By (3), therefore,

$$91 > 9x + 6y + 8z \geq 9(2) + 6(3) + 8z = 36 + 8z$$  \hspace{1cm} (4)

so that,

$$z \leq \frac{55}{8} = 6.875$$  \hspace{1cm} (5)

By (2), however, $z$ is a multiple of 3. Thus, $z$ must equal 6. This gives the following possibilities:

$$(2, 3, 6), (2, 4, 6), (2, 5, 6), (4, 5, 6),$$

We now exclude the last three possibilities as follows.

- If $(x, y, z) = (2, 4, 6)$, then by (1), $m = 45$, which cannot be written as a linear integral combination of 2, 4, and 6.
- If $(x, y, z) = (2, 5, 6)$, then $9x + 6y + 8z = 96 > 91$, violating (3).

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• If \((x, y, z) = (4, 5, 6)\), then \(9x + 6y + 8z = 114 > 91\), violating (3).

Thus, \((x, y, z)\) must be equal to \((2, 3, 6)\) as claimed. This implies that \(m = 42\), which can be written in 35 different ways as a nonnegative linear integral combination of 2, 3 and 6 including \(42 = 0 \cdot 2 + 0 \cdot 3 + 7 \cdot 6\), and \(a = 28\), which can be written in 15 different ways as a nonnegative linear integral combination of 2, 3, and 6 including \(28 = 14 \cdot 2 + 0 \cdot 3 + 0 \cdot 6\).

**CC35.** Evaluate

\[
\lim_{n \to \infty} \frac{1}{n} \sqrt[n]{\frac{(2n)!}{n!}}
\]

(Originally Question 4 from 2001 APICS Math Competition.)

Solved by M. Bataille; C. Curtis; J. G. Heuver; D. E. Manes; P. Perfetti; H. Ricardo; and D. Văcaru. We give the solution of Michel Bataille expanded by the editor.

We re-write each term as 

\[
\sqrt[\sqrt[n]{a_n}], \text{ where } a_n = \frac{(2n)!}{n!n^n}.
\]

We want to determine \(\lim_{n \to \infty} \sqrt[\sqrt[n]{a_n}]\). Since each term in the sequence \(\{a_n\}\) is nonnegative, by the Root Test, if \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \ell\) then \(\lim_{n \to \infty} \sqrt[n]{a_n} = \ell\) so we determine \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \ell\) instead. Observe

\[
\frac{a_{n+1}}{a_n} = \frac{(2n + 2)!}{(n + 1)!(n + 1)^{n+1}} \cdot \frac{n!n^n}{(2n)!} = \frac{(2n + 2)(2n + 1)}{(n + 1)^2} \cdot \frac{1}{(1 + \frac{1}{n})^n},
\]

hence

\[
\frac{a_{n+1}}{a_n} = 2 \cdot \frac{2n + 1}{n + 1} \cdot \frac{1}{(1 + \frac{1}{n})^n}.
\]

Since \(\lim_{n \to \infty} \frac{2n + 1}{n + 1} = 2\) and \(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e\), it follows that

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{4}{e}
\]

and hence this is our desired limit.