SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


The edge lengths of a quadrilateral are $AB = 5$, $BC = 10$, $CD = 11$, $DA = 14$.

(a) If the quadrilateral is cyclic, what is the diameter of its circumcircle?

(b) If we alter the order of the edges, does it affect the answer to (a)?

I. Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

For part (a) note that

$$AB^2 + DA^2 = 5^2 + 14^2 = 221 = 10^2 + 11^2 = BC^2 + CD^2.$$ 

Hence, if we take $BD = \sqrt{221}$, then $ABD$ and $BCD$ will form two right triangles that share their hypotenuse $BD$, which implies that the resulting quadrilateral $ABCD$ has a circumcircle whose diameter is $BD = \sqrt{221}$.

For part (b) the answer is no, altering the order will generally produce a new quadrilateral, but the circumcircle of the new quadrilateral will have the same diameter. To see this, we denote the centre of the circle of part (a) by $O$. Altering the order of the edges is the same as interchanging the triangles $OAB, OBC, OCD, ODA$. No matter how these triangles might be permuted, the four angles at $O$ will still sum to $360^\circ$, and the sides opposite $O$ would form the sides of a new quadrilateral that is still inscribed in the circle whose radius is $OA = OB = OC = OD = \frac{\sqrt{221}}{2}$.

II. Solution by John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA.

We answer both parts together. In the cyclic quadrilateral $ABCD$, let $a = AB = 5$, $b = BC = 10$, $c = CD = 11$, and $d = DA = 14$, and let $s = \frac{1}{2}(a + b + c + d) = 20$ be the semiperimeter. Brahmagupta’s formula gives us the area of the cyclic quadrilateral:


Note that it is a symmetric polynomial in the four variables, so that $A$ is invariant with respect to altering the order of the side lengths. Also the product

$$P = (ac + bd)(ad + bc)(ab + cd) = (55 + 140)(70 + 110)(50 + 154) = 195 \cdot 180 \cdot 204$$

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is a symmetric polynomial and therefore invariant under reordering of the side
lengths. On the *MathWorld* web page for cyclic quadrilaterals [or any other stan-
dard reference] we find the formula $4RA = \sqrt{P}$ involving the circumradius $R$;
consequently, the diameter equals

$$2R = \frac{\sqrt{P}}{2A} = \frac{\sqrt{195 \cdot 180 \cdot 204}}{2 \cdot 90} = \sqrt{13 \cdot 17} = \sqrt{221},$$

and it will not change when the order of the edges is altered.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia
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and the proposer.*

We commonly accept convexity to be part of the definition of a cyclic
quadrilateral. Both solutions show further that the area of a cyclic quadrilateral
will not change when the order of the edges is altered. More precisely, given the four line segments that form a cyclic
quadrilateral, they will, in general, in their six possible orders form three convex quadrilaterals that are not
congruent, yet they will have the same circumradius and the same area. Only the proposer
addressed the corresponding results for crossed quadrilaterals that are inscribed in a circle. The
first solution shows that when the edges are not just rearranged, but are allowed to form a crossed
quadrilateral, those quadrilaterals will still have the same circumcircle. (The formula for $2R$
in the second solution requires convexity; it should not be used for crossed quadrilaterals. Indeed,
the circumradius of a crossed quadrilateral that is inscribed in a circle is generally different from
the common circumradius of its convex mates.)

Most of the submissions were similar to one of the featured solutions, although many
provided more background details. Such details were discussed recently in the solution of the
related problem 2724, which appeared in the March issue [2013 : 148-149].


Show that if $n \geq 2$ is a positive integer then

$$\frac{1}{2} \left[ 1 + \frac{1}{n} \left( 1 - \frac{1}{n} \right) \right]^2 < \left( 1 - \frac{1}{2^3} \right) \left( 1 - \frac{1}{3^3} \right) \cdots \left( 1 - \frac{1}{n^3} \right)$$

holds.

*Solution by Hao Hao Wang and Jerzy Wojdylo, Southeast Missouri State Univer-
sity, Cape Girardeau, Missouri, USA.*

We will prove the claim by induction on $n$.

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First, if \( n \geq 2 \), then the claim holds since
\[
\frac{1}{2} \left[ 1 + \frac{1}{k} \left( 1 - \frac{1}{k} \right) \right]^2 = \frac{25}{32} < \frac{7}{8} = \left( 1 - \frac{1}{2^3} \right).
\]
Assume the claim is true for \( n = k \). So we have
\[
\frac{1}{2} \left[ 1 + \frac{1}{k} \left( 1 - \frac{1}{k} \right) \right]^2 \left( 1 - \frac{1}{(k+1)^3} \right) < \left( 1 - \frac{1}{2^3} \right) \left( 1 - \frac{1}{3^3} \right) \ldots \left( 1 - \frac{1}{k^3} \right) \left( 1 - \frac{1}{(k+1)^3} \right)
\]
and we need to show that the claim is true for \( n = k + 1 \). Multiplying inequality
(1) by \( \left( 1 - \frac{1}{(k+1)^3} \right) \), we obtain
\[
\frac{1}{2} \left[ 1 + \frac{1}{k} \left( 1 - \frac{1}{k} \right) \right]^2 \left( 1 - \frac{1}{(k+1)^3} \right) < \left( 1 - \frac{1}{2^3} \right) \left( 1 - \frac{1}{3^3} \right) \ldots \left( 1 - \frac{1}{k^3} \right) \left( 1 - \frac{1}{(k+1)^3} \right)
\]
Now, we notice that
\[
\frac{1}{2} \left[ 1 + \frac{1}{k} \left( 1 - \frac{1}{k} \right) \right]^2 \left( 1 - \frac{1}{(k+1)^3} \right) - \frac{1}{2} \left[ 1 + \frac{1}{k+1} \left( 1 - \frac{1}{k+1} \right) \right]^2
= \frac{3 - 11k^2 - 8k^3 + 3k^4 + 2k^5}{2k^3(1+k)^4} = \frac{(2k^2 - 8)k^3 + (3k^2 - 11)k^2 + 3}{2k^3(1+k)^4} > 0,
\]
since both \( 2k^2 - 8 \geq 0 \) and \( 3k^2 - 11 > 0 \) as \( k \geq 2 \). Therefore, we have
\[
\frac{1}{2} \left[ 1 + \frac{1}{k+1} \left( 1 - \frac{1}{k+1} \right) \right]^2 < \frac{1}{2} \left[ 1 + \frac{1}{k} \left( 1 - \frac{1}{k} \right) \right]^2 \left( 1 - \frac{1}{(k+1)^3} \right).
\]
Thus, from (2) and (3) the claim is true for \( n = k + 1 \).

This completes the proof of the original inequality for all \( n \geq 2 \).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; MIHAJ-IOAN STOËNESCU, Bischwiller, France; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.


Semi-circles with centres \( O_1 \) and \( O_2 \) are drawn on chords \( AB \) and \( CD \) of a circle \( \Gamma \) such that they are tangent at \( T \). The line through \( O_1 \) and \( O_2 \) intersects \( \Gamma \) at \( E \) and \( F \). If \( O_1A = a \), \( O_2C = b \), \( O_1E = x \) and \( O_2F = y \), show that \( a - b = x - y \).
Solution by several respondents.

By the Intersecting Chords Theorem, we find that $a^2 = x(a + b + y)$ and $b^2 = y(a + b + x)$. Therefore

$$a^2 - b^2 = (x - y)(a + b)$$

from which the result follows.

Solved by MIGUEL AMENGUAL COVAS, Cala Fígura, Mallorca, Spain; AN-ANDUUU Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CAO MINH QUANG, Nguyen Van Cu High School, Vinh Long, Vietnam; DAO THANH OAI, Kien Hoa, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ˇSEFKET ARSLANAGI ´C, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARRO SO CAMPOS, University of Seville, Seville, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; DAO THANH OAI, Kien Xương, Tho Biên, Viet Nam; PRITHWIJIIT DE, Rama Bhabha Centre for Science Education, Mumbai, India; JAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; LEONARD GIUGIUC, Romania; JOHN G. HEUVER, Grande Prairie, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; PANAGIOTE LIOUZAS, Leonardo da Vinci High School, Noci, Italy; RICARD PEIRÓ, IES “Abastos”, Valencia, Spain; CRISTÓBAL SÁNCHEZ-RUBIO, I.B. Penyagolosa, Castellón, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; IRINA STALLION, Northeast Missouri State University, Cape Girardeau, MO, USA; MHI-AIOAN STOÎNESCU, Bischwiller, France; ERCOLE SUPPA, Teramo, Italy; EDMUNDSWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Pietroști, Romania; JACQUES VERNIN, Marseille, France; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Dao proposed the following generalization for which he enclosed a proof by Luis González: Suppose that the circles with diameters $AB$ and $CD$ do not necessarily intersect and that their radical axis meets $EF$ at $T$. Then $TO_1 - TO_2 = EO_1 - EO_2$.

**3754.** [2012: 242, 243] Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Prove that in all scalene triangles $\triangle ABC$ the inequality

$$576\sqrt{3}r^3 < \frac{w_a^2 - w_b^2}{b - a} + \frac{w_b^2 - w_c^2}{c - b} + \frac{w_c^2 - w_a^2}{a - c} < 72\sqrt{3}R^3$$

holds, where $w_a$, $w_b$ and $w_c$ are the lengths of the angle bisectors; $R$ is the radius of the circumcircle; and $r$ is the inradius of $\triangle ABC$.

Solution by Oliver Geipel, Brühl, NRW, Germany.

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The two inequalities do not generally hold. Consider a right triangle with sides \(a = 5t, b = 4t,\) and \(c = 3t\) where \(t\) is a positive real parameter. Its semiperimeter is \(s = 6t.\) Straightforward computations yield

\[
w_a^2 = \frac{4bcs(s-a)}{(b+c)^2} = \frac{288}{49}t^2, \quad w_b^2 = \frac{45}{4}t^2, \quad w_c^2 = \frac{160}{9}t^2,
\]

\[R = \frac{5}{2}t, \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = t.\]

The left inequality thus specializes to \(576\sqrt{3}t^3 < \frac{2624}{147}t,\) which is false when

\[t^2 \geq \frac{2624}{147 \cdot 576\sqrt{3}}t.
\]

The right inequality rewrites as \(\frac{2624}{147}t < 1125\sqrt{3}t^3.\) But this fails when

\[t^2 \leq \frac{2624}{147 \cdot 1125\sqrt{3}}t.
\]

Consequently, both inequalities are not generally valid.

One incorrect solution was received.

Upon closer inspection, the proposer lost a factor part way through his solution (as did the person who sent in the incorrect solution). As a result, the original inequality should have read

\[576\sqrt{3}t^3 < \frac{2624}{147}t,\]

which is less appealing to look at than the original. Using Geulp’s right triangle example in the inequality above yields

\[576\sqrt{3}t^3 < 1524t^3 < 1125\sqrt{3}t^3\]

which is true.

**3755.** [2012 : 242, 244] Proposed by Bill Sands, University of Calgary, Calgary, AB.

Find all real numbers \(a \leq b \leq c \leq d\) which form an arithmetic progression which satisfy the two equations \(a + b + c + d = 1\) and \(a^2 + b^2 + c^2 + d^2 = d.\)

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA; and Titu Zvonaru, Comăneşti, Romania (independently).

Let the four numbers be \(a = m - 3h, b = m - h, c = m + h, d = m + 3h\) where \(h \geq 0.\) The two equations are equivalent to \(4m = 1\) and \(4m^2 + 20h^2 = m + 3h.\) This leads to \(20h^2 = 3h\) which implies that \(h = 0\) or \(h = \frac{3}{20}.\) The two solutions are

\[(a, b, c, d) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(-\frac{1}{5}, \frac{1}{10}, \frac{2}{5}, \frac{7}{10}\right).
\]

Also solved by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Dionne
The proposer points out that if we merely require that $a + b + c + d$ be an integer, then we get exactly two more solutions $(a, b, c, d) = (0, 0, 0, 0), (-9/10, -3/10, 3/10, 9/10)$.

3756. [2012 : 242, 244] Proposed by Michel Bataille, Rouen, France.

Let triangle $ABC$ be inscribed in circle $\Gamma$ and let $M$ be the midpoint of the arc $BC$ of $\Gamma$ not containing $A$. The perpendiculars to $AB$ through $M$ and to $MB$ through $B$ intersect at $K$ and the perpendiculars to $AC$ through $M$ and to $MC$ through $C$ intersect at $L$. Prove that the lines $BC$, $AM$ intersect at the midpoint of $KL$.

Solution by Ricardo Barroso Campos, University of Seville, Seville, Spain, modified by the editor.

Let $N = BC \cap AM$, and $\alpha = \frac{\angle BAC}{2}$. We can assume without loss of generality that $\angle CBA < \angle ACB$ as in the figure (or use directed angles). Because $M$ is the

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midpoint of the arc $BC$,
\[ \alpha = \angle BAM = \angle MAC = \angle MBC. \]

Using the right angles first at the intersection of $KM$ and $AB$ and then at $B$, we have
\[ \angle NMK = \angle AMK = 90^\circ - \alpha \quad \text{and} \quad \angle NBK = \angle MBK - \angle MBC = 90^\circ - \alpha. \]

From $\angle NMK = \angle NBK$ and $\angle MBK = 90^\circ$ we deduce that $KBMN$ is inscribed in a circle whose diameter is $MK$, which makes $\angle MNK = 90^\circ$ also. Analogously, $LCNM$ is cyclic with diameter $ML$ and $\angle MNL = 90^\circ$. Because $MN$ is perpendicular to both $NK$ and $NL$, $N$ must lie on the line $KL$. Also, the right triangles $KNM$ and $LNM$ have corresponding angles of $90^\circ - \alpha$ at their common vertex $M$ and they share the side $MN$; consequently, they are congruent, whence $NK = NL$. That is, $N$ is the midpoint of $KL$.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTÓBAL SÁNCHEZ-RUBIO, I.B. Penyagolosa, Castellón, Spain; EDUARDO SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; JACQUES VERNIN, Marseille, France; MIHAI-IOAN STOENESCU, Bischwiller, France; ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia; I TACHI UCHIHA, Hong Kong, China; JACQUES VERNIN, Marseille, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was one incomplete submission.


Let $A$, $B$, $C$ be the angles (measured in radians), $R$ the circumradius and $r$ the inradius of a triangle. Prove that
\[ \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \leq \frac{9}{2 \pi} \cdot \frac{R}{r}. \]

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Consider the function $f(x) = \ln \left( \frac{\sin \frac{\pi}{x}}{x} \right) = \ln \left( \sin \frac{\pi}{x} \right) - \ln x$, $x \in (0, \pi)$. Straightforward computations show that $f'(x) = \frac{1}{2} \cot \frac{\pi}{x} - \frac{1}{x}$ and
\[ f''(x) = -\frac{1}{4} \csc^2 x + \frac{1}{x^2} = \frac{\sin^2 \left( \frac{\pi}{x} \right) - \frac{1}{4} x^2}{x^2 \sin^2 \left( \frac{\pi}{x} \right)} = \frac{\left( \sin \frac{\pi}{x} + \frac{\pi}{x} \right) \left( \sin \frac{\pi}{x} - \frac{\pi}{x} \right)}{x^2 \sin^2 x}. \]

Since $0 < \sin \frac{\pi}{x} < \frac{\pi}{x}$ for $x \in (0, \pi)$ we have $f''(x) < 0$ so $f$ is concave on $(0, \pi)$. Hence, by Jensen’s Inequality we have
\[ \frac{1}{3} f(A) + \frac{1}{3} f(B) + \frac{1}{3} f(C) \leq f \left( \frac{A + B + C}{3} \right) = f \left( \frac{\pi}{3} \right), \]
so
\[ \frac{1}{3} \left( \ln \left( \frac{\sin \frac{\pi}{A}}{A} \right) + \ln \left( \frac{\sin \frac{\pi}{B}}{B} \right) + \ln \left( \frac{\sin \frac{\pi}{C}}{C} \right) \right) \leq \ln \left( \frac{\sin \frac{\pi}{\frac{\pi}{2}}} {\frac{\pi}{2}} \right) = \ln \left( \frac{3}{2 \pi} \right) \]

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or
\[
\ln \left( \frac{\sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}}{ABC} \right) \leq \ln \left( \frac{3}{2\pi} \right)^3
\]
so
\[
\frac{1}{ABC} \leq \frac{27}{8\pi^3} \frac{1}{\sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2}}.
\] (1)

It is well known that
\[
\sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \sin \frac{C}{2} = \frac{r}{4R}.
\] (2)

From (1) and (2) we then have
\[
\frac{1}{ABC} \leq \frac{27}{2\pi^3} \cdot \frac{R}{r}.
\] (3)

Finally, using (3) and the obvious inequality \(AB + BC + CA \leq \frac{1}{3}(A + B + C)^2\) we have
\[
\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = \frac{AB + BC + CA}{ABC} \leq \frac{(A + B + C)^2}{3ABC} = \frac{9}{2\pi} \cdot \frac{R}{r}
\]
and our proof is complete. Clearly equality holds if and only if \(A = B = C\).

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MARIAN DINCĂ, Bucharest, Romania; NERMIN HODŽIĆ, Dobojica, Bosnia and Herzegovina and SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Arslanagić gave a similar proof and pointed out that (1) is actually inequality 6.59 on p. 188 of the book “Recent Advances in Geometric Inequalities” (Kluwer Academic Publishers, Dordrecht/Boston/London, 1989) by D.S. Mitrinović, J.E. Pečarić and V. Volenec. We decided to give a proof for completeness. Dinca pointed out that since \(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \geq \frac{2\pi}{A+B+C} = \frac{2}{\pi}\) by the AM-HM inequality, the result can be strengthened to a double inequality
\[
\frac{9}{\pi} \leq \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \leq \frac{9}{2\pi} \cdot \frac{R}{r}
\]
or
\[
2r \leq \frac{2\pi r}{9} \left( \frac{1}{A} + \frac{1}{B} + \frac{1}{C} \right) \leq R
\]
which is a refinement of the famous Euler inequality \(2r \leq R\).

3758. [2012 : 242, 244] Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

Given a point \(X\) on the segment \(BC\), construct a point \(A\) such that the incircle of triangle \(ABC\) touches \(BC\) at \(X\), and that the line joining the Gergonne point and the Nagel points of the triangle is parallel to \(BC\).

Composite of solutions by Daniel Văcaru, Pitești, Romania; and by the proposer.

We assume the desired triangle \(ABC\) exists and let \(a = BC\), \(b = CA\), \(c = AB\), and \(s = \frac{1}{2}(a+b+c)\). If \(X\) and \(Z\) are the points where the incircle touches the sides \(BC\) and \(BA\), then (by definition) the cevians \(AX\) and \(CZ\) intersect in the Gergonne point \(G_e\). From the standard properties of incircles we know that

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Applying Menelaus’s theorem to triangle $AX$ we have

$$-1 = \frac{AG_e}{G_eX} \cdot \frac{XC}{CB} \cdot \frac{BZ}{ZA} = \frac{AG_e}{G_eX} \cdot \frac{s - c}{a} \cdot \frac{s - b}{s - a},$$

whence,

$$\frac{AG_e}{G_eX} = \frac{a(s - a)}{(s - b)(s - c)}. \tag{1}$$

We next let $X'$ and $Y'$ be the points where the excircles touch the sides $BC$ and $AC$, so that the cevians $AX'$ and $BY'$ intersect (by definition) in the Nagel point $N_a$. Again we know that $BX' = AY' = s - c$, $CX' = s - b$, and $CY' = s - a$. Applying Menelaus’s theorem to triangle $AX'C$ with transversal $BN_aY'$, we have

$$-1 = \frac{AN_a}{N_aX'} \cdot \frac{X'B}{BC} \cdot \frac{CY'}{Y'A} = \frac{AN_a}{N_aX'} \cdot \frac{-(s - c)}{a} \cdot \frac{s - a}{s - c},$$

whence,

$$\frac{AN_a}{N_aX'} = \frac{a}{s - a}. \tag{2}$$

Because the transversal $G_eN_a$ of the triangle $AX'X'$ is parallel to $XX'$ if and only if $\frac{AG_e}{G_eX} = \frac{AN_a}{N_aX'}$, we deduce from (1) and (2) that $G_eN_a$ is parallel to $BC$ (which contains the segment $XX'$) if and only if

$$(s - a)^2 = (s - b)(s - c). \tag{3}$$

Consequently, we want $s - a$ to be the geometric mean of $s - b$ and $s - c$, a quantity whose construction was given by Euclid. From this we get the lengths $b = (s - a) + (s - c)$ and $c = (s - a) + (s - b)$. Because our argument is reversible, as long as the quantities $a, b, c$ satisfy the triangle inequality, there will be a unique triangle $ABC$ that satisfies (3), whose incircle touches $BC$ at the given point $X$. Here, then, is its construction.

1. Construct the perpendicular to $BC$ at $X$, and call $P$ either point where it intersects the circle whose diameter is $BC$. (Then $PX$ is the geometric mean of $BX = s - b$ and $CX = s - c$; that is, $PX = \sqrt{(s - b)(s - c)} = s - a$.)

2. Call $B'$ and $C'$ the points where the circle with centre $X$ and radius $XP$ intersects the line $BC$, labeled so that $B$ and $C'$ are on the same side of $X$. (Then $BB' = BX + XB' = s - b + s - a = c$ and $C'C = C'X + XC = s - a + s - c = b$.)

3. The desired third vertex $A$ will be either point where the circle with centre $B$ and radius $c = BB'$ intersects the circle with centre $C$ and radius $b = CC'$.

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It remains to verify that the circles in the third step of the construction will always intersect; that is, we must show that the constructed segments \( a, b, c \) satisfy the triangle inequality:

\[
\begin{align*}
  c + b &= BB' + CC' = (BX + XB') + (CX + XC') > BX + XC = a, \\
  a + b &= (BX + XC) + (CX + XC') > BX + XC' = BX + XB' = c, \\
  a + c &= (BX + XC) + (BX + XB') > CX + XB' = CX + XC' = b.
\end{align*}
\]

This proves the existence of the constructed triangle \( ABC \) whose incircle touches \( BC \) at the given point \( X \) and whose sides satisfy equation (3), so that \( G_n \mid \mid BC \) as required.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comănești, Romania.


Given a convex polygon \( A_1A_2 \cdots A_n \) with an interior point \( P \). Let \( a_i = \sum_{j=1}^{n} A_iA_j \). Prove that \( \sum_{i=1}^{n} PA_i < \max_{1 \leq j \leq n} \{ a_j \} \).

Solution by Oliver Geupel, Brühl, NRW, Germany.

The position vector \( \vec{P} \) of a point \( P \) can be expressed as a convex linear combination of the position vectors of the \( A_i \)'s, \( i = 1, 2, \ldots, n \)

\[
\vec{P} = \sum_{i=1}^{n} \lambda_i \vec{A}_i
\]

where \( \lambda_i > 0 \) with \( \sum_{i=1}^{n} \lambda_i = 1 \). Hence,

\[
\vec{A}_i - \vec{P} = \vec{A}_i - \sum_{j=1}^{n} \lambda_j \vec{A}_j = \sum_{j=1}^{n} \lambda_j (\vec{A}_i - \vec{A}_j), \quad i = 1, 2, \ldots, n.
\]

By the triangle inequality, we then have, for each \( i = 1, 2, \ldots, n \),

\[
|PA_i| = \left| \sum_{j=1}^{n} \lambda_j (\vec{A}_i - \vec{A}_j) \right| < \sum_{j=1}^{n} \lambda_j |A_jA_i|.
\]  

[Ed. : For clarity we use \(|XY|\) to denote the distance between points \( X \) and \( Y \), that is, the length of the vector \( \vec{Y} - \vec{X} \).]

Note that the inequality in (1) is strict since the vectors \( \vec{A}_i - \vec{A}_j \) are not collinear.

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Adding the inequality in (1) over \( i = 1, 2, \ldots, n \) we then obtain
\[
\sum_{i=1}^{n} |PA_i| < \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_j |A_j A_i| = \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} |A_j A_i| = \sum_{j=1}^{n} \lambda_j a_j \leq \max_{1 \leq j \leq n} \{a_j\}.
\]
This completes the proof.

Also solved by the proposer.

Geupel remarked that the problem is a generalization of problem 2215 [1997 : 109; 1998 : 121] which dealt with the case \( n = 3 \). This same special problem was also posed in the internet forum Art of Problem Solving. The solution by a solver nicknamed gemath straightforwardly generalizes to his proof featured above. He gave the following reference:


Let \( p \geq 2 \) be an integer. Determine the limit
\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} \sum_{j=1}^{p} \sqrt[\sqrt{n}]{k^j (n+k)^{p-j+1}}.
\]

Solution by Haohao Wang and Jerzy Wojdylo, Southeast Missouri State University, Cape Girardeau, Missouri, USA.

We claim that
\[
\lim_{n \to \infty} \sum_{k \geq 1}^{p} \sum_{j=1}^{p} \sqrt[\sqrt{n}]{k^j (n+k)^{p-j+1}} = \frac{p}{p-1}.
\]
To prove our claim, we note that since \( a + a^2 + \cdots + a^p = \frac{a(1 - a^p)}{1 - a} \), we have
\[
\sum_{j=1}^{p} \sqrt[k^j (n+k)^{p-j+1}] = (n+k)^{(p+1)/p} \left[ \sum_{j=1}^{p} \left( \frac{k}{n+k} \right) \right]^{j/p} = (n+k)^{(p+1)/p} \left( \frac{k}{n+k} \right)^{1/p} \left( 1 - \frac{k}{n+k} \right)^{1/p} = \frac{(n+k)^{(p+1)/p} \left( \frac{k}{n+k} \right)^{1/p} \left( n \right) {n+k} \left( 1 - \left( \frac{k}{n+k} \right)^{1/p} \right)^{1/p}.
\]

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Thus

\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{\sqrt[n]{n}}{ \sqrt{k^j (n + k)^{p-j+1}}} = \lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{\sqrt[n]{n} \left[ 1 - \left( \frac{k}{n+k} \right)^{1/p} \right]}{nk^{1/p}}
\]

\[
= \lim_{n \to \infty} n^{(1-p)/p} \sum_{k=1}^{\infty} \left[ k^{-1/p} - (n + k)^{-1/p} \right]
\]

\[
= \lim_{n \to \infty} n^{(1-p)/p} \sum_{k=1}^{n} k^{-1/p}
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \frac{k}{n} \right)^{-1/p}
\]

\[
= \int_{0}^{1} x^{-1/p} \, dx
\]

\[
= \frac{p}{p-1},
\]
as claimed.

Also solved by JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; ANASTASIOS KOTRONONIS, Athens, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. Three incorrect solutions were received.

Note that this problem is a generalization of problem Q1011 Math. Mag. 84(3), 2011, p. 230.