Harmonic Sets Part 2: Quadrangles and Quadrilaterals

In part 1 we investigated Desargues’s theorem in a projective plane that is represented by the Euclidean plane extended by a line at infinity. We shall now see how that theorem allows us to define harmonic sets. But first we must define the dual notions of quadrangle and quadrilateral.

Four points \( P, Q, R, S \), no three collinear, are the vertices of a complete quadrangle of which the six sides are the lines \( PQ, RS, QR, PS, RP, QS \). The intersections of opposite sides, namely \( A = PQ \cap RS, \ C = QR \cap PS, \ U = RP \cap QS \), are called diagonal points. See Figure 1.

Four lines \( p, q, r, s \), no three concurrent, are the sides of a complete quadrilateral of which the six vertices are the points \( p \cap q, r \cap s, q \cap r, p \cap s, r \cap p, q \cap s \). The joins of opposite vertices, namely \( a = (p \cap q)(r \cap s), \ c = (q \cap r)(p \cap s), \ u = (r \cap p)(q \cap s) \), are called diagonal lines. See Figure 2.

Four collinear points \( A, B, C, D \) are said to form a harmonic set if there is a quadrangle of which two opposite sides pass through \( A \) and two other opposite sides through \( C \), while one of the remaining sides passes through \( B \) and the other through \( D \). We say that \( B \) and \( D \) are harmonic conjugates with respect to \( A \) and \( C \).

\[ \begin{align*}
A = PQ \cap RS, & \quad C = QR \cap PS, \\
U = RP \cap QS, & \quad a = (p \cap q)(r \cap s), \quad c = (q \cap r)(p \cap s), \\
& \quad u = (r \cap p)(q \cap s),
\end{align*} \]

Figure 1: The complete quadrangle \( PQRS \).

Figure 2: The complete quadrilateral \( pqrs \).
C, a relationship we abbreviate by \( H(AC, BD) \). Desargues's theorem then tells us that \( H(AC, BD) \) if and only if \( H(BD, AC) \): As shown in Figure 3, triangles \( QBR \) and \( PSD \) are perspective from \( A \), so the points \( C = QR \cap PS \) and \( U = BR \cap DS \) must be collinear with the new point \( V = BQ \cap DP \); thus \( B \) and \( D \) are diagonal points of the quadrangle \( VQUP \) while \( A \) and \( C \) lie on the remaining sides, whence \( H(BD, AC) \), as claimed. Similarly, any classical projective geometry text would show how Desargues’s theorem justifies the definition of a harmonic set; in other words, it implies that the harmonic conjugate of \( B \) with respect to \( A \) and \( C \) is uniquely defined and does not depend of the choice of quadrangle. (The details: Given any three collinear points \( A, B, \) and \( C \), choose \( R \) to be any point not on the line \( AB \), and take \( S \neq A, R \) arbitrarily on \( RA \). Then \( P = RB \cap SC, Q = AP \cap RC, \) and \( D = QS \cap AC \); moreover, a different choice of \( R \) and \( S \) produces the same point \( D \) by three applications of Desargues’s theorem.)

![Figure 3: \( H(AC, BD) \) if and only if \( H(BD, AC) \).](image)

A harmonic set of lines through a point is defined by the dual of the definition of a harmonic set of points. In Figure 2, for example, in the quadrilateral \( pqrs \) we have \( H(ac, bd) \) for the four lines through the point \( R \), which is the intersection of the diagonals \( a \) and \( b \). It turns out that for any line \( \ell \) and any point \( O \) not on \( \ell \), the points \( A, B, C, D \) form a harmonic set of points on \( \ell \) if and only if the lines \( OA, OB, OC, OD \) form a harmonic set of lines through \( O \), as is easily seen by superimposing Figure 1 on Figure 2: let the points in Figure 2 have the labels \( P = p \cap r, Q = s \cap p, R = a \cap c, S = s \cap r, \) etc. so that \( A, B, C, D \) are the points where \( q \) intersects the corresponding lines through \( R \). As we have just seen, we could have used any point of the plane not on \( q \) to play the role of \( R \) in locating the harmonic conjugate of \( B \) with respect to \( A \) and \( C \). As a consequence, we deduce that if \( A, B, C, D \) and \( A', B', C', D' \) are two quadruples of collinear points that are situated so that \( AA', BB', CC', DD' \) all pass through a point \( O \), then \( H(AC, BD) \) implies \( H(A'C', B'D') \). We say that these two harmonic sets are related by a \textit{perspectivity with center} \( O \). With the help of Desargues’s theorem one can prove that any two harmonic sets are related by a sequence of perspectivities [1, Section
Exercise 1. Prove that the two sides of the diagonal triangle of a quadrangle that meet in any diagonal point are harmonic conjugates with respect to the two sides of the quadrangle through that point.

Exercise 2. Prove that if $B, C, D$ and $B', C', D'$ are triples of collinear points on distinct lines that intersect at $A$ while both $H(AC, BD)$ and $H(AC', B'D')$, then the lines $BB', CC', DD'$ are concurrent.

Exercise 3. Use the isosceles trapezoid $PQRS$ shown in Figure 4 to verify two familiar examples of harmonic sets.

- If $D$ is the point at infinity of the line $AC$, then it is the harmonic conjugate of the midpoint of the segment $AC$ with respect to $A$ and $C$.
- When the lines $b$ and $d$ are perpendicular, then they are harmonic conjugates with respect to $a$ and $c$ if and only if they bisect the angles formed by $a$ and $c$.

Exercise 4. Part 1 of this series explained how a perspective collineation is determined by a center, an axis, and the image of one point different from the center but not on an axis. Prove that if the central collineation has period two (such as a reflection in a point or a line of the Euclidean plane), then any point $P$ is the harmonic conjugate of its image point $P'$ with respect to the center and the point where $PP'$ intersects the axis.

Exercise 5. If $PQR$ is a triangle and $H(AA', QR)$ and $H(BB', RP)$, then prove that $P$ and $Q$ are harmonic conjugates with respect to

$$C = AB' \cap BA' \quad \text{and} \quad C' = AB \cap A'B'.$$

The theory of harmonic sets can be derived algebraically using coordinates and cross ratios. For this, one needs the notion of directed distance: on a given

line we arbitrarily choose one direction as the positive direction and the other as the negative direction. A segment $AB$ on the line will then be considered positive or negative according as the direction from $A$ to $B$ is the positive or negative direction. Because it will always be clear from the context, the same symbol $AB$ will be used to denote the line $AB$ (extended infinitely in both directions) as well as the distance directed from $A$ to $B$. Because we will be working in the extended Euclidean plane, when $I$ is the point at infinity of the line $AB$ we define the ratio $\frac{AI}{IB} = -1$, which is consistent with the notion of the ratio $\frac{AP}{PB}$ approaching $-1$ as $P$ tends to infinity in either direction along the line $AB$. We can now define the cross ratio $(AB; CD)$ of four points on a line of the Euclidean plane to be

$$(AB; CD) = \frac{AC \cdot BD}{AD \cdot BC}.$$ 

Note that with the points listed in a different order, one might get a different cross ratio. An important theorem in most classical projective geometry texts tells us that cross ratios are preserved by perspectivities; another asserts that when four points lie on a line in the order $A, B, C, D$, then $B$ and $D$ are harmonic conjugates with respect to $A$ and $C$ if and only if $(AC; BD) = -1$. Likewise, the lines $OA, OB, OC, OD$ form a harmonic set if the cross ratio of their slopes equals $-1$.

**Exercise 6.** Repeat Exercise 3 using coordinates for part (a) and slopes for part (b).

**References**