CONTEST CORNER
SOLUTIONS

CC16. In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum. Given the magic square shown with $a, b, c, x, y, z > 0$, determine the product $xyz$ in terms of $a, b$ and $c$.

(Originally question A6 from the 2011 Canadian Senior Mathematics Contest.)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Richard I. Hess, Rancho Palos Verdes, CA, USA; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We present the solution of Manes.

Let $S$ denote the sum of each row, column, and diagonal. Taking the sum along the first row and the minor diagonal gives

\[
\log(abx) = S = \log(xyz),
\]

$xz = abx$, and 

\[ (1) \]

\[
yz = ab.
\]

\[ (2) \]

Similarly, considering the third column with the main diagonal gives us

\[
xc = ay
\]

and taking the first column with the second row gives us

\[
az = yc.
\]

\[ (4) \]

Combining equations (2) and (4) gives us

\[
z = \sqrt{bc}.
\]

\[ (5) \]

We are now able to solve for the value of $xyz$.

\[
xyz = abx
\]

from (1),

\[
= \frac{a^2by}{c}
\]

from (3),

\[
= \frac{a^3bz}{c^2}
\]

from (4),

\[
= \frac{a^3b^2}{c^2}
\]

from (5).

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CC17. A line with slope \( m \) meets the parabola \( y = x^2 \) at \( A \) and \( B \). If the length of segment \( AB \) is \( \ell \) what is the equation of that line in terms of \( \ell \) and \( m \)? (Inspired by question \# 8 from the 2010 Manitoba Mathematical Competition.)

Solved by Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Richard I. Hess, Rancho Palos Verdes, CA, USA; Mihaï-Ioan Stoïnescu, Bischwiller, France; Daniel Văcaru, Pitești, Romania; and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We present the solution of Coiculescu.

Let \( y = mx + n \) be the equation of the line intersecting the parabola \( y = x^2 \).

The points of intersection, \( A \) and \( B \) are:

\[
A = (x_1, y_1) = \left( \frac{m - \sqrt{m^2 + 4n}}{2}, x_1^2 \right),
\]

\[
B = (x_2, y_2) = \left( \frac{m + \sqrt{m^2 + 4n}}{2}, x_2^2 \right).
\]

The square of the length of the segment \( AB \) is

\[
\ell^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2.
\]

We have that \( x_2 - x_1 = \sqrt{m^2 + 4n} \) and \( x_2 + x_1 = m \), so we see that:

\[
\ell^2 = (y_2 - y_1)^2 + (x_2 - x_1)^2
= (x_2 - x_1)^2(x_2 + x_1)^2 + (x_2 - x_1)^2
= (m^2 + 4n)(m^2 + 1).
\]

Solving for \( n \) yields

\[
n = \frac{1}{4} \left( \frac{\ell^2}{m^2 + 1} - m^2 \right),
\]

so the equation of the line is

\[
y = mx + \frac{1}{4} \left( \frac{\ell^2}{m^2 + 1} - m^2 \right).
\]

CC18. The left end of a rubber band \( e \) meters long is attached to a wall and a slightly sadistic child holds on to the right end. A point-sized ant is located at the left end of the rubber band at time \( t = 0 \), when it begins walking to the right along the rubber band as the child begins stretching it. The increasingly tired ant walks at a rate of \( 1/(\ln(t + e)) \) centimeters per second, while the child uniformly stretches the rubber band at a rate of one meter per second. The rubber band is infinitely stretchable and the ant and child are immortal. Compute the time in seconds, if it exists, at which the ant reaches the right end of the rubber band. (Originally question \# 8 from the 2012 Stanford Math Tournament, team test.)
Solved by Richard I. Hess, Rancho Palos Verdes, CA, USA.

The initial length of the band is $100e$ and the length as a function of time is $L(t) = 100(t + e)$, both in centimetres. In the time interval $dt$ at time $t$ the ant covers a fraction of the band’s total length equal to

$$f(t) = \frac{dt}{100(t + e)\ln(t + e)}.$$  

Thus, the total fraction of the band covered in time $T$ is

$$F(T) = \int_{0}^{T} \frac{dt}{100(t + e)\ln(t + e)} = \int_{e}^{T+e} \frac{du}{100u\ln u} = \frac{1}{100} [\ln(\ln(T+e)) - \ln(\ln e)].$$  

The time at which $F(T) = 1$ is when $\ln(\ln(T+e)) = 100 \Rightarrow \ln(T+e) = e^{100}$, thus the ant reaches the end of the band at $T = e^{100} - e$ seconds.

**CC19.** Evaluate

$$\frac{1}{3} + \frac{1}{3+4+\frac{1}{5+\frac{1}{6+\frac{1}{\cdots+2013}}}} + \frac{1}{1+\frac{1}{2+\frac{1}{4+\frac{1}{5+\frac{1}{\cdots+2013}}}}}.$$  

*Inspired by question # 6 from the 2005 University of Waterloo, Bernoulli Trials.*

Solved by Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Richard I. Hess, Rancho Palos Verdes, CA, USA; and Mihai-Ioan Stoicescu, Bischwiller, France. We present the solution of Coiculescu and Stoicescu.

Let $a = 2 + \frac{1}{3+4+\frac{1}{5+\frac{1}{6+\frac{1}{\cdots+2013}}}}$. Then we see that,

$$\frac{1}{3} + \frac{1}{3+4+\frac{1}{5+\frac{1}{6+\frac{1}{\cdots+2013}}}} = \frac{1}{1+a} + \frac{1}{1+\frac{1}{a}} = \frac{1}{1+a} + \frac{a}{1+a} = 1.$$  

**CC20.** When the Math Club advertises an “$(M, N)$ sock hop”, this means that the DJ has been instructed that the $M$th dance after a fast dance must be a slow dance, while the $N$th dance after a slow dance must be a fast dance. (All dances are slow or fast; the DJ avoids the embarrassing ones where nobody is quite sure what to do.) For some values of $M$ and $N$ this means that the dancing must end early and everybody can start in on the pizza; for other values the dancing can in principle go on forever. For which ordered pairs $(M, N)$ is there no upper bound to the number of dances?

*Originally question # 7 from the 2008 Science Atlantic Math Competition.*

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We regard the sequence of dances as a sequence \( \{x_k\} \) of 1s and 0s, with a 1 representing a fast dance and a 0 representing a slow dance. Write \( M = 2^r \cdot m \) and \( N = 2^s \cdot n \), where \( m \) and \( n \) are odd. We claim that there is an infinite sequence satisfying the requirements if and only if \( r = s \).

We note that the \( M \th \) dance before a fast dance is a slow dance, since the alternative leads immediately to a contradiction. Likewise, the \( N \th \) dance before a slow dance is a fast dance.

Suppose that \( r < s \), and suppose the sequence of dances is infinite. Choose a positive integer \( p \) such that \( x_p = 0 \). Then we have the following chains of implications:

\[
\begin{align*}
x_p = 0 & \Rightarrow x_{p+N} = 1 \Rightarrow x_{p+M+N} = 0 \Rightarrow x_{p+M} = 1 \Rightarrow x_{p+2M} = 0 \\
& \Rightarrow \cdots \Rightarrow x_{p+4M} = 0 \Rightarrow \cdots \Rightarrow x_{p+(2^s-r \cdot n)M} = 0
\end{align*}
\]

and

\[
\begin{align*}
x_p = 0 & \Rightarrow x_{p+N} = 1 \Rightarrow x_{p+M+N} = 0 \Rightarrow x_{p+M+2N} = 1 \Rightarrow x_{p+2N} = 0 \\
& \Rightarrow \cdots \Rightarrow x_{p+4N} = 0 \Rightarrow \cdots \Rightarrow x_{p+mN} = 1.
\end{align*}
\]

But,

\[
p + (2^s-r \cdot n)M = p + (2^s-r \cdot n)(2^r \cdot m) = p + (2^s \cdot n) \cdot m = p + mN,
\]

so we have a contradiction. Thus the sequence must be finite.

If instead \( r > s \), the sequence again terminates. The proof starts with a positive integer \( q \) such that \( x_q = 1 \) and proceeds as before.

If \( r = s = 0 \), so that \( M \) and \( N \) are odd, each of the sequences,

\[
1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \cdots \quad \text{or} \quad 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \cdots
\]

is an \((M, N)\) sock hop.

If \( r = s \geq 1 \), we define a sequence as follows. The first \( 2^r \) terms of the sequence can be chosen arbitrarily from \( \{0, 1\} \). For each \( j \in \{1, 2, 3, \ldots, 2^r\} \), the subsequence \( \{x_{j+k \cdot 2^r}\}_{k=1}^{\infty} \) is chosen to be an alternating sequence of 1s and 0s. The resulting sequence is an \((2^r \cdot m, 2^r \cdot n)\) sock hop.

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