SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The editor would like to acknowledge PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; whose solution to problem 3718 was overlooked. The editor apologizes sincerely for the oversight.


Given the triangle $ABC$ and two isogonal cevians $AA'$, $AA''$, call $B', C'$ the orthogonal projections of $B, C$ on $AA'$ and $B'', C''$ the orthogonal projections of $B, C$ on $AA''$. If $P = B'C'' \cap C'B''$ and $Q = B'B'' \cap C'C''$, show that $P$ lies on line $BC$ and $Q$ lies on the altitude through $A$. Dedicated to the memory of Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.

Solution by Michel Bataille, Rouen, France.

(a) We first show that $Q$ is on the altitude from $A$.

Let $F$ be the foot of the altitude from $A$. Because $AF \perp FB$, $AB' \perp B'B$, and $AB'' \perp B''B$, the circle $\Gamma_b$ with diameter $AB$ passes through $F, B', B''$. Similarly, the circle $\Gamma_c$ with diameter $AC$ passes through $F, C', C''$. If we use $\angle(\ell, \ell')$ to denote the angle directed from $\ell$ to $\ell'$ (through which $\ell$ must be rotated in the positive direction in order to become parallel to, or to coincide with $\ell'$), we have, modulo $\pi$,

$$\angle(B'A, B'B'') = \angle(BA, BB''), \quad \text{that is,} \quad \angle(B'C'', B'B'') = \angle(AB, AA'') + \frac{\pi}{2}$$
[where the angles on the left of each equality are identical while those on the right are oppositely oriented angles in the same right triangle $\angle B''B''C''$]; in the same way

$$\angle(C''B'', C''C') = \angle(CA, CC') = \angle(AC, AA') + \frac{\pi}{2}.$$ 

Thus,

$$\angle(B'C', B'B'') - \angle(C''C', C''B'') = \angle(AB, AA'') + \angle(AC, AA') = 0,$$

where the last equality follows from the assumption that $AA'$ and $AA''$ are isogonal. As a consequence, $B', C', B'', C''$ are concyclic, say on the circle $\Gamma$. The power of $Q$ with respect to $\Gamma$ is $\frac{QB^2}{2} = \frac{QC^2}{2}$, which means that the power of $Q$ with respect to $\Gamma_c$ equals the power of $Q$ with respect to $\Gamma_c$. Thus, $Q$ is on the radical axis of the two circles, which is just the altitude $AF$.

(b) We now prove that $P$ lies on the line $BC$.

Because $B'C'$ intersects $B''C''$ at $A$, while $B''C', B'C''$ intersect at $P$ and $C'C'', B'B''$ intersect at $Q$, the triangle $APQ$ is self-conjugate with respect to $\Gamma$. In particular, $AQ$, as the polar of $P$ with respect to $\Gamma$ is perpendicular to $MP$, where $M$ is the centre of $\Gamma$. Since $AQ \perp BC$, the lines $PM$ and $BC$ would coincide (in which case $P$ would lie on $BC$), if $M$ were on $BC$. To prove the latter, consider the inversion $I$ with centre $A$ such that $I(B') = C'$. We then have $I(B'') = C''$ and the lines $B'B'', C'C''$ invert into $\Gamma_c$, $\Gamma_b$, respectively. It follows that $I$ exchanges $Q$ and $F$, and that the circle with diameter $AQ$ inverts into $BC$, the perpendicular to $AQ$ at $F$. Now, $A$ and $Q$ being conjugate with respect to $\Gamma$, the circle with diameter $AQ$ is orthogonal to $\Gamma$ and so its image $BC$ under $I$ is a diameter of $\Gamma$; that is, $M \in BC$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany (property of point $\Gamma$ only); JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

GeupeL observed that much of our problem follows from paragraphs 63 and 633 of Nathan Altshiller Court’s College Geometry, 2nd ed., 1952; in particular, Court provides an alternative proof that the four orthogonal projections $B', B'', C', C''$ of $B$ and $C$ upon the sides of $\angle A'\AA''$ lie on a common circle whose centre is the midpoint of $BC$. Here is Court’s elementary proof that the centre $M$ of $\Gamma$ is the midpoint of $BC$. Because $B'C'$ is a chord of $\Gamma$, its perpendicular bisector contains $M$; moreover, because this perpendicular bisector along with $BB'$ and $CC'$ are three parallel lines that intercept equal segments on the transversal $BC''$, these lines must likewise intersect the transversal $BC$ in $B, C$, and the midpoint of the segment $BC''$. The same argument applied to the segment $B''C''$ yields a second line (not parallel to the first) that also contains both $M$ and the midpoint of $BC$, which implies that these two points must coincide.


Prove that

$$\left(\frac{1}{4} - 4 \cos^2 \frac{2\pi}{17} \cos^2 \frac{8\pi}{17}\right) \left(\frac{1}{4} - 4 \cos^2 \frac{3\pi}{17} \cos^2 \frac{5\pi}{17}\right) + 4 \cos \frac{2\pi}{17} \cos \frac{3\pi}{17} \cos \frac{5\pi}{17} \cos \frac{8\pi}{17} = 0.$$ 

I. Solution by Itachi Uchiha, Hong Kong, China.

Let \( x = \cos \frac{2\pi}{17} \cos \frac{8\pi}{17} \) and \( y = \cos \frac{3\pi}{17} \cos \frac{5\pi}{17} \).

It is required to show that
\[
(1 - 16x^2)(1 - 16y^2) = -64xy.
\]

We begin by showing that
\[
1 + 4x - 4y + 16xy = 0.
\]

Repeated use of the product-to-sum conversion formula and the relation
\[
\cos \left( \frac{34k\pi}{17} \right) = \cos \frac{2k\pi}{17},
\]
yields that
\[
1 + 4x - 4y + 16xy = 1 + 8 \sum_{k=1}^{8} \cos \frac{2k\pi}{17} = 1 + \sum_{k=1}^{8} \left( \cos \frac{2k\pi}{17} + \cos \frac{34 - 2k\pi}{17} \right)
\]
\[
= \sum_{k=0}^{16} \cos \frac{2k\pi}{17} = \text{Re} \sum_{k=0}^{16} \zeta^k = 0,
\]
where \( \zeta = \cos \frac{2\pi}{17} + i \sin \frac{2\pi}{17} \) is a primitive seventeenth root of unity. Therefore
\[
(1 - 4x)(1 + 4y) = 1 - 4x + 4y - 16xy = 2 - (1 + 4x - 4y + 16xy) = 2
\]
and
\[
(1 + 4x)(1 - 4y) = (1 + 4x - 4y + 16xy) - 32xy = -32xy.
\]
Multiplying these two equations yields the desired result.

II. Solution by the proposer.

Let \( \zeta = \cos \frac{2\pi}{17} + i \sin \frac{2\pi}{17} \),

\[
a = 4 \cos \frac{2\pi}{17} \cos \frac{8\pi}{17} = 2 \left( \cos \frac{6\pi}{17} + \cos \frac{10\pi}{17} \right) = \zeta^3 + \zeta^{14} + \zeta^5 + \zeta^{12}
\]
and
\[
b = 4 \cos \frac{3\pi}{17} \cos \frac{5\pi}{17} = 2 \left( \cos \frac{2\pi}{17} + \cos \frac{8\pi}{17} \right) = \zeta + \zeta^{16} + \zeta^4 + \zeta^{13}.
\]

It is required to show that
\[
(1 - a^2)(1 - b^2) + 4ab = 0.
\]
For easier computation, introduce \( c = \zeta^6 + \zeta^{11} + \zeta^7 + \zeta^{10} \) and \( d = \zeta^2 + \zeta^{15} + \zeta^8 + \zeta^9 \). Since \( a + b + c + d = -1 \) and \( ab = 2b + c + d = b - a - 1 \), therefore

\[
(1 - a^2)(1 - b^2) + 4ab = (1 - a)(1 - b)(1 + a)(1 + b) + 4ab \\
= (1 + ab - a - b)(1 + ab + a + b) + 4ab \\
= (b - a - a - b)(b - a + a + b) + 4ab \\
= (-2a)(2b) + 4ab = 0.
\]

**III. Solution using elements of solutions from Brian D. Beasley, Presbyterian College, Clinton, SC, USA; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA.**

Let \( a_k = \cos(k\pi/17) \) and \( x = a_2 = \cos(2\pi/17) \). The left side of the identity is equal to

\[
\frac{1}{16}[(1 + 16a_2a_3a_5)^2 - 16(a_2a_8 - a_3a_5)^2] \\
= \frac{1}{16}(1 + 16a_2a_3a_5 - 4a_2a_8 + 4a_3a_5)(1 + 16a_2a_3a_5 + 4a_2a_8 - 4a_3a_5).
\]

Since

\[
a_2a_8 = a_2[2a_4^2 - 1] = a_2[2(2a_2^2 - 1)^2 - 1] = x(8x^4 - 8x^2 + 1)
\]

and

\[
a_3a_5 = a_2 + a_8 = 8x^4 - 8x^2 + x + 1,
\]

the final factor on the right side is equal to

\[
1 + 8x(8x^4 - 8x^2 + 1)(8x^4 - 8x^2 + 1) \\
+ 4x(8x^4 - 8x^2 + 1) - 2(8x^4 - 8x^2 + x + 1) \\
= 512x^9 - 1024x^7 + 64x^6 + 672x^5 - 80x^4 - 160x^3 + 24x^2 + 10x - 1 \\
= (2x - 1)f(x),
\]

where

\[
f(x) = 256x^8 + 128x^7 - 448x^6 - 192x^5 + 240x^4 + 80x^3 - 40x^2 - 8x + 1.
\]

With \( T_{17}(x) \) denoting the Chebyshev polynomial of degree 17, we have that

\[
1 = \cos 2\pi = T_{17}(x) \\
= 65536x^{17} - 278528x^{15} + 487424x^{13} - 452608x^{11} \\
+ 239360x^9 - 71808x^7 + 11424x^5 - 816x^3 + 17x
\]

so that \( f(x) = 0 \) and the desired result follows.

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW.

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Germany; KEE-WAI LAU, Hong Kong, China; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and TITU ZVONARU, Comănești, Romania.

Geupel and the Angelo State University trio showed that the left side was equal to

\[
\frac{17}{16} \left(1 - 2 \sum_{k=1}^{8} (-1)^{k+1} a_k\right),
\]

which vanished because the sum was shown to be equal to 1/2. Lau followed the strategy of the third solution and showed that the left side is equal to

\[
\frac{(3a)^n}{(b+1)(c+1)} + \frac{(3b)^n}{(c+1)(a+1)} + \frac{(3c)^n}{(a+1)(b+1)} \geq \frac{27}{16}.
\]


Let \(a, b, c\) be positive real numbers such that \(a + b + c = 1\). If \(n\) is a positive integer, prove that

\[
\frac{(3a)^n}{(b+1)(c+1)} + \frac{(3b)^n}{(c+1)(a+1)} + \frac{(3c)^n}{(a+1)(b+1)} \geq \frac{27}{16}.
\]

Solution by Itachi Uchiha, Hong Kong, China.

By the Power Mean Inequality followed by the AM-GM Inequality, we have

\[
(3a)^n(a+1) + (3b)^n(b+1) + (3c)^n(c+1)
\]

\[
= \frac{1}{3} \left[(3a)^{n+1} + (3b)^{n+1} + (3c)^{n+1}\right] + 3 \cdot \frac{1}{3} \left[(3a)^n + (3b)^n + (3c)^n\right]
\]

\[
\geq \left(\frac{3a + 3b + 3c}{3}\right)^{n+1} + 3 \cdot \frac{3a + 3b + 3c}{3}
\]

\[
= 1 + 3 = 4 = \frac{27}{16} \left(\frac{4}{3}\right)^3 = \frac{27}{16} \left(\frac{a + 1 + b + 1 + c + 1}{3}\right)^3
\]

\[
\geq \frac{27}{16} (a+1)(b+1)(c+1).
\]

Divide both sides by \((a+1)(b+1)(c+1)\) and the result follows. Clearly we have equality if and only if \(a = b = c = \frac{1}{3}\).

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; DIONNE BAILEY, ELISIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RADOUAN BOUKHARFANE, Polytechnique de Montréal, QC; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MARIAN DINCĂ, Bucharest, Romania; OLEH FAYNSHTEYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DANIEL VĂCARU, Pitești, Romania; HAOHAO WANG and YANPING XIA, Southeast Missouri State University, Cape Girardeau, MO, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

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I. Solution by George Apostolopoulos, Messolonghi, Greece.

Label the sides of our quadrilateral $ABCD$ by $AB = a = 7$, $BC = b = 8$, $CD = c = 4$, $DA = d = 1$, and the unknown diagonals $AC = x$, $BD = y$. Ptolemy’s theorem tells us that the diagonals satisfy

$$xy = ac + bd, \quad \text{and} \quad \frac{y}{x} = \frac{ab + cd}{ad + bc};$$

that is

$$xy = 7 \cdot 4 + 8 \cdot 1 = 36 \quad \text{and} \quad \frac{y}{x} = \frac{7 \cdot 8 + 4 \cdot 1}{7 \cdot 1 + 8 \cdot 1} = \frac{20}{13}.$$

Multiplying gives us $y = 12\sqrt{\frac{5}{13}} = \frac{12}{13}\sqrt{65} \approx 7.442$; division yields $x = 3\sqrt{\frac{5}{8}} = \frac{3}{8}\sqrt{65} \approx 4.837$.

A cyclic quadrilateral is, by definition, convex; on the other hand, it turns out to be interesting to find the lengths of the diagonals of a crossed quadrilateral inscribed in a circle with edge lengths 7, 8, 4, 1, in that order; this is the quadrilateral $A'B'CD'$ in the accompanying figure. The vertices lie clockwise in the order $A'D'B'C$, so that the resulting convex quadrilateral has sides $A'D' = 1$, $D'B = y$, $BC = 8$, $CA' = x$, and diagonals $CD = 4$, $AB = 7$. Ptolemy’s theorem now implies that

$$4 \cdot 7 = xy + 8 \cdot 1 \quad \text{and} \quad \frac{4}{7} = \frac{1 \cdot y + x \cdot 8}{1 \cdot x + y \cdot 8};$$

so that the diagonals of $A'B'CD'$ are

$$x = 5 \cdot \sqrt{\frac{5}{13}} = \frac{5}{13}\sqrt{65} \approx 3.101, \quad y = 4 \sqrt{\frac{13}{5}} = \frac{4}{5}\sqrt{65} \approx 6.450.$$
II. Solution by Mihaï-Ioan Stoënescu, Bischwiller, France.

Le théorème des cosinus dans les triangles \( \triangle DAB \) et \( \triangle BCD \) nous dit que:

\[
BD^2 = DA^2 + AB^2 - 2 \cdot DA \cdot AB \cos \angle A, \quad \text{et} \quad BD^2 = BC^2 + CD^2 - 2 \cdot BC \cdot CD \cos \angle C.
\]

Or les angles \( A \) et \( C \) sont supplémentaires donc \( \cos \angle A = - \cos \angle C = t \). En remplaçant par les valeurs données, on tire que

\[
1 + 49 - 14t = 64 + 16 + 64t,
\]

d’où \( t = -\frac{9}{13} \). Par conséquent, \( BD^2 = 50 + \frac{70}{13} = \frac{720}{13} \).

Il vient que \( BD = \sqrt{\frac{720}{13}} \).

Le même théorème dans les triangles \( \triangle ABC \) et \( \triangle CDA \) donne que \( \cos \angle D = -\frac{4}{5} \) et

\[
AC = \sqrt{\frac{720}{13}}.
\]

Also solved by ŠEFKET ARSLANAGIČ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELsie CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD L. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; DIMITRIOS KOUKAKIS, Kilkis, Greece; KEE-WAI LAU, Hong Kong, China; PANAGIOTE LIGOURAS, Leonardo da Vinci High School, Noci, Italy; SALEM MALIKIČ, student, Simon Fraser University, Burnaby, BC; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John’s, NL; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Pitești, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Almost all submissions resembled one of the featured solutions; some used directly the known formulas for the diagonals of cyclic quadrilateral (which follow immediately from the two forms of Ptolemy’s theorem used in the first solution). Those formulas are readily found on the internet (just google “cyclic quadrilateral”) or in college geometry texts. Compare the related problem 3751 [2012 : 241, 243] which calls for the circumradius of an inscribed quadrilateral in terms of its side lengths. Although there is a familiar formula also for this, Woo observed that the proposer chose the lengths (in both problems) quite carefully: note that \( 7^2 + 4^2 = 8^2 + 1^2 = 65 \).

In other words, the edges can be rearranged to form a quadrilateral composed of a pair of right triangles that share a hypotenuse of length \( \sqrt{65} \). It is an easy exercise to show (without resorting to algebra) that any convex cyclic quadrilateral with sides of length \( 1, 4, 7, \) and \( 8 \) in any order would have the same value \( \sqrt{65} \) as its circumradius. As a bonus in our problem, because one of those quadrilaterals has a diagonal that is a diameter of the circle (the line \( BB' \) in the figure), the crossed quadrilateral from the first solution is also inscribed in the same circle.


Prove that the sequence of nonzero real numbers, \( x_1, x_2, \ldots \), is a geometric progression if and only if it satisfies the recurrence relation

\[
x_n x_{n+1} = \sum_{k=1}^{n} x_k x_{n+1-k}, \quad n = 1, 2, \ldots
\]
Composite of similar solutions by all solvers.

If $x_1, x_2, \ldots$ is a geometric progression, then for some constant $r > 0$, we have $x_n = r^{n-1}x_1$ for all $n \in \mathbb{N}$. Hence

$$\sum_{k=1}^{n} x_k x_{n+1-k} = \sum_{k=1}^{n} r^{k-1} x_1 r^{n-k} x_1 = \sum_{k=1}^{n} x_1 (r^{n-1} x_1) = nx_1 x_n.$$ 

Conversely, suppose $\sum_{k=1}^{n} x_k x_{n+1-k} = nx_1 x_n$ for all $n \in \mathbb{N}$. We prove by induction that for all $n \in \mathbb{N}$,

$$x_n = r^{n-1}x_1, \quad \text{where} \quad r = \frac{x_2}{x_1}. \tag{1}$$

Since (1) is clearly true for $n = 1, 2$, we assume that it holds for $i = 1, 2, \ldots, n$ for some $n \geq 2$, then we have

$$(n + 1)x_1 x_{n+1} = \sum_{k=1}^{n+1} x_k x_{n+2-k} = 2x_1 x_{n+1} + \sum_{k=2}^{n} x_k x_{n+2-k}$$

so

$$x_{n+1} = \sum_{k=2}^{n} r^{k-1} x_1 r^{n+1-k} x_1 = \sum_{k=2}^{n} r^{n} x_1^2 = (n-1)r^n x_1^2$$

from which it follows that $x_{n+1} = r^n x_1$ completing the induction and the proof.

Solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; D. M. BĂȚINÊTU-GIURGIU, Bucharest, Romania and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; RADOUAN BOUKHARFANE, Polytechnique de Montréal, QC; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOSEPH DiMURO, Biola University, La Mirada, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Pitești, Romania; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.


Let $A, B, C, s, r, R$ represent the angles (measured in radians), the semi-perimeter, the inradius and the circumradius of a triangle, respectively. Prove that

$$\left(\frac{A}{B} + \frac{B}{C} + \frac{C}{A}\right)^3 \geq \frac{2s^2}{Rr}.$$ 

Composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and the proposer.

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By the AM-GM Inequality, we have
\[ \frac{A}{B} + \frac{A}{B} + \frac{B}{C} \geq 3\sqrt[3]{\frac{A^2}{BC}} = \frac{3A}{\sqrt[3]{ABC}}. \]

Similarly, \( \frac{B}{C} + \frac{B}{C} + \frac{C}{A} \geq \frac{3B}{\sqrt[3]{ABC}} \) and \( \frac{C}{A} + \frac{C}{A} + \frac{A}{B} \geq \frac{3C}{\sqrt[3]{ABC}}. \) Adding the three inequalities above we have
\[ \frac{A}{B} + \frac{B}{C} + \frac{C}{A} \geq \frac{A + B + C}{\sqrt[3]{ABC}} = \frac{\pi}{\sqrt[3]{ABC}}. \quad (1) \]

In [1], it was proved that
\[ \frac{abc}{ABC} \geq \left( \frac{2s}{\pi} \right)^3 \quad (2) \]
where \( a, b, c \) denote the lengths of the sides of the given triangle. Since it is well known that \( abc = 4Rrs \), it follows from (1) and (2) that
\[ \left( \frac{A}{B} + \frac{B}{C} + \frac{C}{A} \right)^3 \geq \frac{\pi^3}{ABC} \geq \frac{(2s)^3}{abc} = \frac{8s^3}{4Rrs} = \frac{2s^2}{Rr}, \]
and the proof is complete. It is easy to see that equality holds if and only if the triangle is equilateral.

References


Also solved by MARIAN DINÇĂ, Bucharest, Romania; EDMUND SWYLAN, Riga, Latvia; and PETER Y. WOO, Biola University, La Mirada, CA, USA;

3727. [2012 : 105, 107] Proposed by J. Chris Fisher, University of Regina, Regina, SK.

Let \( ABCD \) and \( AECF \) be two parallelograms with vertices \( E \) and \( F \) inside the region bounded by \( ABCD \). Prove that line \( BE \) bisects segment \( CF \) if and only if \( BF \) meets \( AD \) in a point \( G \) that satisfies
\[ \frac{DA}{DG} = \frac{BF}{FG}. \]

I. Solution by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India.

Observe that if either of \( E \) or \( F \) were to lie on the diagonal \( BD \), then so would the other, in which case we would have neither \( BE \) bisecting \( CF \) nor \( DA/DG = BF/FG \).
Thus let us assume that $E$ does not lie on $BD$, so that $BE$ will intersect $AD$ in a point, call it $A'$, for which $GBA'$ is a proper triangle. Further, define the points $X$ and $Y$ to be the intersections of $BE$ with $CF$ and $CD$, respectively. From the symmetry of the given parallelograms about their common centre we conclude that $DF$ and $BE$ are parallel, whence in triangle $CFD$ we see that $X$ is the midpoint of $CF$ if and only if $Y$ is the midpoint of $CD$. Moreover, in triangle $GBA'$ we have $\frac{DA}{DA'} = \frac{BF}{FG}$. It follows that the desired equality $\frac{DA}{DA'} = \frac{BF}{FG}$ would hold if and only if $DA = DA'$. Our assumption that the point $E$ is inside $ABCD$ implies that $A \neq A'$, and we could have $DA = DA'$ if and only if $D$ were the midpoint of $AA'$. This in turn is equivalent to the triangles $DA'Y$ and $CBY$ being symmetric about their common vertex $Y$, which occurs if and only if $Y$ is the midpoint of $CD$, which (as we remarked earlier) is equivalent to $BE$ intersecting $CF$ in its midpoint $X$.

\[ BE \text{ bisects } CF \iff X \in BE \iff Y \in BE \iff Z \in DF \text{ (by symmetry)} \]

\[ \iff \frac{BF}{FG} \cdot \frac{GD}{DA} \cdot \frac{AZ}{ZB} = -1 \]

\[ \iff \frac{DA}{DG} = \frac{BF}{FG} \text{ (as directed distances).} \]

Observe that if we interpret notation such as $PQ$ to represent the length of the line segment joining $P$ to $Q$ (instead of representing the distance directed from $P$ to $Q$), and if $D$ were between $G$ and $A$, then $\frac{DA}{DG} = \frac{BF}{FG}$ would imply that $F$ must be situated between $G$ and $B$ on a side of $\Delta GAB$ with $DF || AB$. But that would place $F$ on the line $CD$ and, thus, not inside the parallelogram $ABCD$ as prescribed.
We conclude that the result holds whether or not we interpret distances as being directed.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; and the proposer. There was one incomplete submission.

With a little care, either solution shows that the result continues to hold when \( F \) is chosen anywhere in the plane of \( ABCD \) as long as it is not on the line \( CD \). Note further that this comment together with solution II implies that our problem is equivalent to the theorem, If \( F \) is any point in the plane of \( \triangle ABD \) that is not on the parallel to \( AB \) through \( D \), and \( G \) is the point where \( BF \) meets \( AD \), then \( F \) lies on the median through \( D \) if and only if \( G \) satisfies
\[
\frac{DA}{DG} = \frac{BF}{FG}.
\]

3728. [2012 : 105, 107] Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Given a continuous function \( f : [0, \frac{\pi}{2}] \to \mathbb{R} \) that satisfies
\[
\int_0^{\frac{\pi}{2}} \left( (f(x))^2 - 2f(x)(\sin x - \cos x) \right) \, dx = 1 - \frac{\pi}{2},
\]
show that
\[
\int_0^{\frac{\pi}{2}} f(x) \, dx = 0.
\]

Composite of many submitted solutions.

Observe that
\[
\int_0^{\frac{\pi}{2}} [f(x) - (\sin x - \cos x)]^2 \, dx
= \int_0^{\frac{\pi}{2}} (f(x))^2 \, dx - 2\int_0^{\frac{\pi}{2}} f(x)(\sin x - \cos x) \, dx + \int_0^{\frac{\pi}{2}} (\sin x - \cos x)^2 \, dx
= \left(1 - \frac{\pi}{2}\right) + \int_0^{\frac{\pi}{2}} (1 - \sin 2x) \, dx = \left(1 - \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - 1\right) = 0,
\]
whence \( f(x) = \sin x - \cos x \) and \( \int_0^{\frac{\pi}{2}} f(x) \, dx = 0 \).

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; D.M. BATINETU-GIURGIU, Bucharest and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; RADOUAN BOUKHARFANE, Polytechnique de Montréal, QC; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLEH FAYNShteYN, Leipzig, Germany; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JAYSON K.T. SMITH, Southeastern Missouri State University, Cape Girardeau, MO; ALBERT STADLER, Herrliberg, Switzerland; ITACHI UCHIHA, Hong Kong, China; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

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The main variant in the solutions was due to Boukharfane who replaced \( \sin x - \cos x \) by \( \sqrt{2}\sin(x - \pi/4) \).


If \( a, b, c \) are the side lengths of a triangle, prove that

\[
\frac{b + c}{a^2 + bc} + \frac{c + a}{b^2 + ca} + \frac{a + b}{c^2 + ab} \leq \frac{3(a + b + c)}{ab + bc + ca}.
\]

Solution by the proposer, expanded slightly by the editor.

The given inequality is equivalent, in succession, to

\[
\sum_{\text{cyclic}} \left( \frac{1}{a} - \frac{b + c}{a^2 + bc} \right) \geq \left( \sum_{\text{cyclic}} \frac{1}{a} \right) - \frac{3(a + b + c)}{ab + bc + ca},
\]

\[
\sum_{\text{cyclic}} \frac{a^2 + bc - ab - ac}{a(a^2 + bc)} \geq \frac{ab + bc + ca}{abc} - \frac{3(a + b + c)}{ab + bc + ca},
\]

\[
\sum_{\text{cyclic}} \frac{(a - b)(a - c)}{a(a^2 + bc)} \geq \frac{(ab + bc + ca)^2 - 3abc(a + b + c)}{abc(ab + bc + ca)}
= \frac{\sum \limits_{\text{cyclic}} (b^2c^2 + bca^2 - b^2ca - bc^2a)}{abc(ab + bc + ca)},
\]

\[
\sum_{\text{cyclic}} (a - b)(a - c) \left( \frac{1}{a(a^2 + bc)} - \frac{1}{a(ab + bc + ca)} \right) \geq 0,
\]

\[
\sum_{\text{cyclic}} \frac{(a - b)(a - c)(b + c - a)}{(a^2 + bc)(ab + bc + ca)} \geq 0,
\]

\[
\sum_{\text{cyclic}} \frac{(a - b)(a - c)(b + c - a)}{a^2 + bc} \geq 0.
\]

(1)

Now, without loss of generality, we can assume that \( a \geq b \geq c \). Since \( a, b, \) and \( c \) are the side lengths of a triangle, we have

\[
ca - c^2 - ab + b^2 = (b^2 - c^2) - a(b - c) = (b - c)(b + c - a) \geq 0
\]

so \( c(a - c) \geq ab - b^2 = b(a - b) \). Hence

\[
a - c \geq \frac{b(a - b)}{c}.
\]

(2)

Using (2) we have, since $a - b \geq 0$ and $a - c \geq 0$,

$$\sum \frac{(a - b)(a - c)(b + c - a)}{a^2 + bc} \geq \frac{(b - c)(b - a)(c + a - b)}{b^2 + ca} + \frac{(c - a)(c - b)(a + b - c)}{c^2 + ab}$$

$$\geq \frac{(b - c)(b - a)(c + a - b)}{b^2 + ca} + \frac{b(a - b)(b - c)(a + b - c)}{c(c^2 + ab)}$$

$$= (a - b)(b - c)\left( \frac{b(a - b)}{c(c^2 + ab)} - \frac{c + a - b}{b^2 + ca} \right)$$

$$\geq (a - b)(b - c)\left( \frac{b(c + a - b)}{c(c^2 + ab)} - \frac{c + a - b}{b^2 + ca} \right)$$

$$= (a - b)(b - c)(c + a - b)(b^3 - c^3) \geq 0,$$

which establishes (1) and completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. Three of these solutions were either computer aided or by complicated argument using Schur’s Inequality. Lau gave a proof using calculus together with Euler’s Theorem ($2r \leq R$) and some other known results. The solution given by the proposer and featured above is the only elementary one. There was also an incorrect solution and a solution making claims with no justifications.


Points $D, E$ and $F$ are the feet of the perpendiculars from some point $P$ in the plane to the lines $BC, CA$ and $AB$ determined by the sides of an equilateral triangle $ABC$. Prove that the cevians $AD, BE, CF$ are concurrent (or parallel) if and only if at least one of $D, E$ or $F$ is a midpoint of its side.

Composite of solutions by Peter Y. Woo, Biola University, La Mirada, CA, USA; and by Titu Zeonaru, Comănești, Romania.

Without loss of generality we assume that the sides of the equilateral triangle have length 2 so that if $BD = \alpha, CE = \beta,$ and $AF = \gamma$, then $DC = 2 - \alpha, EA = 2 - \beta$, and $FB = 2 - \gamma$. Consequently,

$$\alpha^2 - (2 - \alpha)^2 = PB^2 - PC^2,$$

$$\beta^2 - (2 - \beta)^2 = PC^2 - PA^2,$$

$$\gamma^2 - (2 - \gamma)^2 = PA^2 - PB^2.$$

Adding them, we get

$$\alpha + \beta + \gamma = 3. \quad (1)$$

Ceva’s theorem tells us that $AD, BE, CF$ are concurrent or parallel if and only if

$$\alpha \beta \gamma = (2 - \alpha)(2 - \beta)(2 - \gamma),$$

or

$$2\alpha \beta \gamma = 8 - 4(\alpha + \beta + \gamma) + 2(\alpha \beta + \beta \gamma + \gamma \alpha).$$

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By (1) this is equivalent to \( \alpha \beta \gamma - (\alpha \beta + \beta \gamma + \gamma \alpha) + 2 = 0 \), or

\[
(\alpha - 1)(\beta - 1)(\gamma - 1) = 0.
\]

Hence, \( AD, BE, CF \) are concurrent or parallel if and only if either \( \alpha = 1 \), or \( \beta = 1 \), or \( \gamma = 1 \); in other words, if and only if \( D \) is the midpoint of \( BC \), or \( E \) is the midpoint of \( CA \), or \( F \) is the midpoint of \( AB \).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; EDMUND SWYLAN, Riga, Latvia; and the proposer.

Geupel reminds us that problem 2508 [2000 : 46; 2001 : 58-61; 2003 : 402] dealt with points having the property of \( P \); in particular, any point \( D \) on the line joining the vertices \( B \) and \( C \) of an arbitrary triangle \( ABC \) determines 0, 1, 2, or infinitely many positions of a point \( P \) whose pedal triangle \( DEF \) has vertices with the property that \( AD, BE, CF \) are concurrent or parallel. The current problem shows that when the triangle is equilateral, there are always exactly two candidates for \( P \) for each position of \( D \) on the line \( BC \) except for the midpoint of the segment \( BC \) (in which case \( P \) could be any point on the perpendicular to \( BC \) through \( D \)).