THE OLYMPIAD CORNER
No. 311

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Electronic submissions are preferable, with each solution contained in a separate file. Files should be named using the convention \texttt{LastName\_FirstName\_OCProblemNumber} (example \texttt{Doe\_Jane\_OC1234.tex}). It is preferred that readers submit a \texttt{BT\_\_E\_X} file and a pdf file for each solution, although other formats, such as Microsoft Word, are also accepted. Readers are invited to email solutions and contests to the editor at \texttt{crux-olympiad@cms.math.ca}. Submissions by regular mail are also accepted and should be sent to the address inside the back cover. Name(s) of solver(s) with affiliation, city, and country should appear on each solution, and each solution should start on a separate page.

To facilitate their consideration, solutions to the problems should be received by the editor by \textbf{1 July 2014}, although solutions received after this date will also be considered until the time when a solution is published.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions' section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet, of l’ Université Saint-Boniface in Winnipeg, for translations of the problems.

\textbf{OC121.} Prove that for all positive real numbers $x, y, z$ we have
\[
\sum_{\text{cyclic}} (x + y)\sqrt{(z + x)(z + y)} \geq 4(xy + yz + zx).
\]

\textbf{OC122.} We define a sequence $f_n(x)$ of functions by
\[
f_0(x) = 1, f_1(x) = x, \quad (f_n(x))^2 - 1 = f_{n-1}(x)f_{n+1}(x), \quad \text{for } n \geq 1.
\]
Prove that for every $n$, $f_n(x)$ is a polynomial with integer coefficients.

\textbf{OC123.} Let $p$ be prime. Find all positive integers $n$ for which, whenever $x$ is an integer such that $x^n - 1$ is divisible by $p$, then $x^n - 1$ is also divisible by $p^2$.

\textbf{OC124.} Find all triples $(a, b, c)$ of positive integers with the following property: for every prime $p$, if $n$ is a quadratic residue (mod $p$), then $an^2 + bn + c$ is also a quadratic residue (mod $p$).

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**OC125.** ABC is an acute angle triangle with $\angle A > 60^\circ$ and $H$ as its orthocenter. $M, N$ are two points on $AB, AC$ respectively, such that $\angle HMB = \angle HNC = 60^\circ$. Let $O$ be the circumcenter of triangle $HMN$. Let $D$ be a point on the same side of $BC$ as $A$ such that $\triangle DBC$ is an equilateral triangle. Prove that $H, O, D$ are collinear.

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**OC121.** Démontrer l’inégalité suivante, pour tout $x, y, z$ nombres réels positifs :

$$\sum_{cyclic} (x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx).$$

**OC122.** Une suite de fonctions $f_n(x)$ est définie par

$$f_0(x) = 1, f_1(x) = x, (f_n(x))^2 - 1 = f_{n-1}(x)f_{n+1}(x), \text{ pour } n \geq 1.$$ 

Démontrer que pour tout $n$, $f_n(x)$ est un polynôme avec coefficients entiers.

**OC123.** Soit $p$ un nombre premier. Déterminer tous les entiers positifs $n$ pour lesquels, aussitôt que $x$ est un entier tel que $x^n - 1$ est divisible par $p$, il en découle que $x^{n-1} - 1$ est divisible par $p^2$.

**OC124.** Déterminer tous les triplets $(a, b, c)$ formés d’entiers positifs et vérifiant la propriété suivante : pour tout nombre premier $p$, si $n$ est un résidu quadratique (mod $p$), alors $an^2 + bn + c$ est aussi un résidu quadratique (mod $p$).

**OC125.** ABC est un triangle à angles aigus avec $\angle A > 60^\circ$, dont l’orthocentre est $H$. $M$ et $N$ sont deux points sur $AB$ et $AC$ respectivement, tels que $\angle HMB = \angle HNC = 60^\circ$. Soit $O$ le centre du cercle circonscrit du triangle $HMN$. Soit aussi $D$ un point qui se trouve sur le même côté de $BC$ que $A$ et tel que $\triangle DBC$ est un triangle équilatéral. Démontrer que $H, O$ et $D$ sont colinéaires.
OLYMPIAD SOLUTIONS

OC61. 46 squares of a 9 × 9 grid are coloured red. Prove that we can find a 2 × 2 square on the grid which contains at least 3 red squares.

(Originally question 2 from the 2011 Singapore National Olympiad.)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania.

We give the solution of Zvonaru.

Let \( a_{i,j} \), for \( 1 \leq i, j \leq 9 \), represent the squares of the grid. We consider \( a_{ij} = 1 \) if the square is red and \( a_{ij} = 0 \) otherwise. For each \( 1 \leq i \leq 9 \) we denote

\[
    r_i = \sum_{j=1}^{9} a_{ij}.
\]

If \( r_1 \leq 5 \), we have

\[
    (r_2 + r_3) + (r_4 + r_5) + (r_6 + r_7) + (r_8 + r_9) = 46 - r_1 \geq 41,
\]

hence, there exists a \( i \in \{2, 4, 6, 8\} \) such that

\[
    r_i + r_{i+1} \geq 11.
\]

If \( r_3 \leq 5 \), we have

\[
    (r_1 + r_2) + (r_4 + r_5) + (r_6 + r_7) + (r_8 + r_9) = 46 - r_3 \geq 41,
\]

hence, there exists a \( i \in \{1, 4, 6, 8\} \) such that

\[
    r_i + r_{i+1} \geq 11.
\]

Repeating the argument, we can deduce that if one of \( r_1, r_3, r_5, r_7 \) or \( r_9 \) is less or equal than 5, then there exists some \( 1 \leq i \leq 8 \) such that

\[
    r_i + r_{i+1} \geq 11.
\]

Moreover, since \( r_1, r_9 \leq 9 \) we have

\[
    (r_1 + r_2) + (r_2 + r_3) + ... + (r_8 + r_9) = 2 \cdot 46 - r_1 - r_9 \geq 74.
\]

Hence, there exists some \( 1 \leq i \leq 8 \) such that \( r_i + r_{i+1} \geq 10 \). Moreover, either one of \( r_i \) or \( r_{i+1} \) is at least 6, or \( r_i = r_{i+1} = 5 \). In the second case by the first part of the solution, there exists some \( 1 \leq j \leq 8 \) such that

\[
    r_j + r_{j+1} \geq 11.
\]

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Thus, it suffices to prove the problem under the assumption that there exists an $i$ so that
\[ r_i + r_{i+1} \geq 10 \text{ and } r_i \geq 6. \]
If $r_i + r_{i+1} \geq 11$, then the conclusion is clear, so without loss of generality we can assume that $r_i + r_{i+1} = 10$.

For convenience, we assume that $r_{i+1} \geq 6$. As $r_i \leq 4$, there exists some $i \in \{1, 3, 5, 7, 9\}$ such that
\[ a_{1i} + a_{2i} \leq 1. \]
Removing the squares $a_{1i}$ and $a_{2i}$ from the first two columns, we are left with four $2 \times 2$ squares. As these 4 squares have $r_1 + r_2 - (a_{1i} + a_{2i}) \geq 9$ red squares, we are done.

**OC62** Let $A, B, C, D$ be four non-coplanar points in space. The segments $AB, BC, CD$ and $DA$ are tangent to the same sphere. Prove that their four points of tangency are coplanar.

(Originally question 3 from the 2011 Spanish Olympiad, Day 1.)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comănești, Romania. We give the solution of Bataille.

Let segments $AB, BC, CD, DA$ touch the sphere at $R, S, T, U$, respectively, and let
\[ x = AR = AU; \quad y = BR = BS; \quad z = CS = CT \quad \text{and} \quad t = DT = DU. \]
Denoting by $\mathbf{M}$ the vector from a fixed point to $M$, let $I$ be the point determined by
\[ m \mathbf{I} = yz \mathbf{A} + zt \mathbf{B} + tx \mathbf{C} + xyz \mathbf{D}, \]
where
\[ m = yz + zt + tx + xyz. \]
Then
\[ m \mathbf{I} = zt \left( y \mathbf{A} + x \mathbf{B} \right) + xy \left( t \mathbf{C} + z \mathbf{D} \right) = zt (y + x) \mathbf{R} + xy (t + z) \mathbf{T}. \]

Because $zt (y + x)$ and $xy (t + z)$ are positive and sum to $m$, it follows that $I$ lies on the segment $RT$. Similarly,
\[ m \mathbf{I} = yz \left( t \mathbf{A} + x \mathbf{D} \right) + tx \left( z \mathbf{B} + y \mathbf{C} \right) = yz (t + x) \mathbf{U} + tx (z + y) \mathbf{S}. \]
showing that $I$ lies on the segment $US$ as well. Thus, the lines $US$ and $RT$ are concurrent at $I$, and determine a plane containing $R, S, T, U$. The result follows.
OC63. Prove that there exists a perfect square so that the sum of its digits is 2011.

(Originally question 4 from 2011 Finland Math Olympiad.)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; R. Laumen, Deurne, Belgium; Daniel Văcaru, Pitești, Romania; and Titu Zvonaru, Comănești, Romania. We give the solution of Zvonaru.

We denote by $s(m)$ the sum of digits of $m$. It is easy to see that $s(m^2) = 0, 1, 4, 7 \pmod{9}$. We will prove now that if $n = 0, 1, 4, 7 \pmod{9}$ there exists a perfect square with the sum of digits $n$.

We have

$$a = \left(\frac{999}{9}..9\right)^2 = \frac{999}{9}..9 \times 0.1 \Rightarrow s(a) = 9k,$$

$$a = \left(\frac{999}{9}.1\right)^2 = \frac{999}{9}..9 \times 0.81 \Rightarrow s(a) = 9k + 1,$$

$$a = \left(\frac{999}{9}.2\right)^2 = \frac{999}{9}..9 \times 0.64 \Rightarrow s(a) = 9k + 4,$$

$$a = \left(\frac{999}{9}.4\right)^2 = \frac{999}{9}..9 \times 0.36 \Rightarrow s(a) = 9k + 7.$$

For $n = 2011$ we have

$$\left(\frac{999}{9}.2\right)^2 = \frac{999}{9}..9 \times 0.64$$

and thus

$$s\left(\frac{999}{9}..9 \times 0.64\right) = 221 \times 9 + 8 + 4 + 221 \times 0 + 6 + 4 = 2011.$$

Văcaru’s, Geupel’s and Laumen’s solution is

$$\left(\frac{999}{9}.7\right)^2 = \frac{999}{9}..9 \times 0.9.$$

Several readers noted that the more general problem

Determine all the possible values for the sum of digits of a perfect square.

and variations of it have appeared in several other competitions and have been discussed in books, such as T. Andreescu, R. Gelca - “Mathematical Olympiad Challenges”.

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OC64. Find all integer solutions of the equation

\[ n^3 = p^2 - p - 1 \]

where \( p \) is prime.

(Originally question 5 from the 2011 Italy Math Olympiad.)

Solved by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zeonaru, Comăneci, Romania. There were also one incomplete and one incorrect solution. We give the solution of Geupel.

It is easy to verify that \((n, p) = (1, 2)\) and \((n, p) = (11, 37)\) are solutions of the equation. We prove that there are no other solutions.

Suppose that the integer \( n \) and the prime \( p \) are solution to the given equation. Then we have \( 0 < n < p \) and

\[ p(p - 1) = (n + 1)(n^2 - n + 1), \]

which implies that the prime number \( p \) divides either \( n + 1 \) or \( n^2 - n + 1 \). We consider both cases in succession.

First assume that \( p \) divides \( n + 1 \). Then there is a positive integer \( a \) such that \( n + 1 = ap \). Since \( n < p \), we have \( a = 1 \). Using (1), we obtain

\[ n = p - 1 = n^2 - n + 1, \]

that is

\[ (n - 1)^2 = 0. \]

It follows that \((n, p) = (1, 2)\).

Second assume that \( p \) divides \( n^2 - n + 1 \). Then there is a positive integer \( b \) such that \( n^2 - n + 1 = bp \). Applying (1), we obtain

\[ p - 1 = b(n + 1). \]

It follows that

\[ n^2 - n + 1 = bp = b(b(n + 1) + 1). \]

We successively obtain \( n^2 - (b^2 + 1)n - (b^2 + b - 1) = 0 \) and

\[ n = \frac{1}{2}(b^2 + 1 \pm \sqrt{b^4 + 6b^2 + 4b - 3}). \]

Thus, the number \( b^4 + 6b^2 + 4b - 3 \) is a perfect square. This is not valid for \( b = 1 \) or \( b = 2 \). For \( b = 3 \), the equation (3) and (2) yield \((n, p) = (11, 37)\). For \( b \geq 4 \), we have

\[ (b^2 + 3)^2 = b^4 + 6b^2 + 9 < b^4 + 6b^2 + 4b - 3 \]
\[ < b^4 + 6b^2 + 4b - 3 + 2(b - 1)^2 + 17 = (b^2 + 4)^2. \]

This is a contradiction, which completes the second case.

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OC65. Let $ABC$ be a triangle. $F$ and $L$ are two points on the side $AC$ such that $AF = LC < AC/2$. If $AB^2 + BC^2 = AL^2 + LC^2$ find $\angle FBL$.

(Originally question 4 from 2011 Morocco National Olympiad, Grade 11.)

Solved by Michel Bataille, Rouen, France; Matei Coiculescu, East Lyme High School, East Lyme, CT, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Daniel Văcărău, Pitești, Romania; and Titu Zvonaru, Comănești, Romania. We give the solution of Bataille.

The median from vertex $B$ in $\triangle ABC$ is also the median from $B$ in $\triangle FBL$. It follows that

$$2(AB^2 + BC^2) - AC^2 = 2(BL^2 + BF^2) - FL^2.$$  

Let $\theta = \angle FBL$. Since

$$BL^2 + BF^2 - FL^2 = 2BF \cdot BL \cdot \cos(\theta),$$

and

$$AB^2 + BC^2 = AL^2 + LC^2,$$

(1) can be rewritten as

$$2(AL^2 + LC^2) - (AL + LC)^2 = 4BF \cdot BL \cdot \cos(\theta) + FL^2,$$

or

$$(AL - LC)^2 = 4BF \cdot BL \cdot \cos(\theta) + FL^2.$$ 

Since $AL - LC = AL - AF = FL$ we finally get

$$4BF \cdot BL \cdot \cos(\theta) = 0.$$ 

Thus $\theta = 90^\circ$. 

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