MAYHEM SOLUTIONS

M504. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Inside a right triangle with sides 3, 4, 5, two equal circles are drawn that are tangent to one another and to one leg. One circle of the pair is tangent to the hypotenuse. The other circle of the pair is tangent to the other leg. Determine the radii of the circles in both cases.

[*Ed.: This problem was mistakenly republished as problem 3724 [2012 : 105, 106]. It was noticed and later replaced with a new problem [2012 : 194, 194]. This was not quick enough for a few Crux readers who managed to send in their solutions before the problem was retracted. Several of the submissions generalized the problem much like the featured solution [2012 : 309]. The generalization to an arbitrary triangle is presented below.*]

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

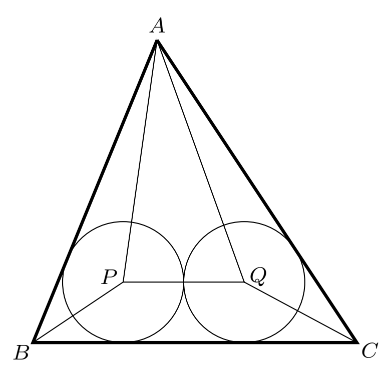
More generally, we will consider an arbitrary triangle with two circles of equal radii tangent to one another and to one of the sides with each of the circles being tangent to one of the remaining two sides. Consider the figure below. Denote the radii of the circles by r, the altitude of the triangle by h, and the area of triangle ABC by $K = ah/2$. Let $BC = a, AC = b$, and $AB = c$. Triangle AQC has area $br/2$, triangle APB has area $cr/2$, trapezoid $BPQC$ has area $(a + 2r)r/2$, and triangle APQ has area $(h - r)r$. Therefore

$$\begin{aligned} K &= br/2 + cr/2 + (a + 2r)r/2 + (h - r)r \\ &= (a/2 + b/2 + c/2 + 2K/a)r, \end{aligned}$$

so

$$r = \frac{2K}{a + b + c + 4K/a},$$

where K can be determined by Heron's formula.



If triangle ABC is a right triangle with hypotenuse c, then $K = ab/2$ and $r = ab/(a+3b+c)$. For the original problem, taking $a = 4, b = 3$, and $c = 5$ gives $r = 2/3$ and taking $a = 3, b = 4$, and $c = 5$ gives $r = 3/5$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; RADOUAN BOUKHARFANE, Polytechnique de Montréal, PQ; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA*; EDMUND SWYLAN, Riga, Latvia; DANIEL VĂCARU, Piteşti, Romania; ITACHI UCHIHA, Hong Kong, China; and TITU ZVONARU, Comăneşti, Romania*. The asterisk (*) indicates a generalization of the right triangle problem.*

M506. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB; and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

We are trying to create a set of positive integers, that each can be formed using their own digits only, along with any mathematical operations and/or symbols that are familiar to you. Each expression must include at least one symbol/operation; the number of times a digit appears is the same as in the number itself. For example, $1 = \sqrt{1}$, $36 = 6 \times 3!$ and $121 = 11^2$. All valid contributions will be acknowledged.

Ed.: Since the publication of solutions to this problem, the editor has received several pieces of correspondence.

Richard I. Hess, Rancho Palos Verdes, CA, USA, sent the editor expressions for all values from 7 to 105 inclusive, which included some values not represented before such as $10 = .1^{-(0!)}$ *and* $27 = \left\lceil \sqrt{((\sqrt{2+7})!)!} \right\rceil$. *Hess also included more than* 300 *other solutions in the range* 1000 - 2000 *including several formulas good for ranges such as* $1200 — 1209 = (0!/.2)!/.1 +$ *units digit. He indicated to the editor by email that he has recently completed* 1100 *more solutions in the range* 2000 - 9999 *which are on the way by regular mail as this is being prepared.*

Stan Wagon, Macalester College, St. Paul, MN, USA, noted that if one used the double factorial function,

$$n!! = \begin{cases} n \times (n-2) \times \cdots \times 5 \times 3 \times 1 & n > 0 \text{ is odd} \\ n \times (n-2) \times \cdots \times 6 \times 4 \times 2 & n > 0 \text{ is even} \\ 1 & n = 0 \end{cases}$$

we could get the nicer solution $15 = 1(5!!)$. *Further, Wagon suggests the subfactorial function* $!n$ *which counts the number of derangements of* n *objects (a derangement of* n *objects is a permutation where no object is in its original position). Using this function we get* $1\left\lfloor \sqrt{!6} \right\rfloor = 16$. *He then proceeded to give expressions for* 19, 23, 43 *and* 82 *using various combinations of the ceiling function, square roots and subfactorials.*

As mentioned when the original solutions were published, Hess conjectured that all numbers could be reached using combinations of the factorial, square root, floor and ceiling functions. Wagon noticed this conjecture and wrote some Mathe-

matica programs to find solutions. Wagon quickly reached expression for all numbers from 1 *through* 100*, for example*

$$7 = \left\lceil \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\left\lceil \sqrt{7!} \right\rceil !}}}}}}} \right\rceil .$$

Correspondence went on between Hess, Wagon, Al Zimmermann and Joseph DeVincentis. A summary of their findings are given below in an email from Wagon to the editor.

> First, the problem was conjectured by Donald Knuth some years ago using Floor only (and not Ceiling), so the discussion below will use that setup.
>
> Second, Joseph DeVincentis observed that it can be stated in this form: Can one start at 3 and reach any integer by iterating factorial, square root, and floor? This is the form Knuth had as well. This suffices, since any number can be brought down to 3 or 4 by repeating square roots. And getting from 4 to 3 can be done by:
>
> $$\left\lfloor \sqrt{\sqrt{\left\lfloor \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{(4!)!}}}}} \right\rfloor !}} \right\rfloor = 3$$
>
> So if we can get from 3 to n, then we can get from n to n.
>
> So now the computational problem is to start at 3 and start building up a table of what is reachable. Of course, the factorials become unwieldy. Dick Hess programmed his own Stirling formula approximation; Mathematica has LogGamma built-in. So that allows one to go quite far. There are issues of how to store numbers, whether to work with integers or machine reals, whether to store the paths found for every n, and so on. While it seems as if roundoff error would be a killer, in fact the situation is quite stable so that even working with machine reals seems adequate to get reliable paths (which can be checked after the fact using ultra-high precision). So the current state is as follows.
>
> I obtained paths from 3 to every number up to 2000. Dick Hess has gotten up to 40389 by taking 199 generations and keeping track of numbers (i.e., logfactorial numbers, so in fact he is checking up to 48 million factorial) under 48 million. So the first unresolved case is 40390. I have not found much online about whether others have checked this conjecture to these realms. In any case, the evidence is quite strong that all numbers can be reached from 3.
>
> Thanks to Joseph, Al Zimmermann, and Dick Hess for help with this intriguing problem.

The problem has inspired Wagon, as shown in the Macalester College Problem of the Week *problem number* 1171 Four Unary Functions *which can be viewed at*

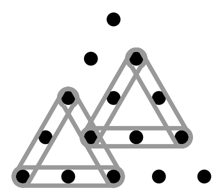
`http://mathforum.org/wagon/fall13/p1171.html`,

with solutions (large file!) at

`http://mathforum.org/wagon/current_solutions/s1171.html`.

At the time of publication, Wagon has extended the his solutions to 131 110, *inclusive.*

M513. *Proposed by the Mayhem Staff.*

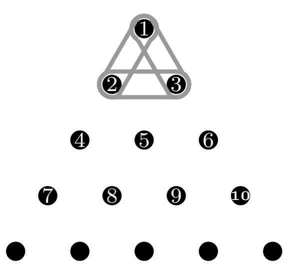


An equilateral triangular grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form equilateral triangles, two different equilateral triangles two centimetres on each side are shown in the diagram. How many different equilateral triangles are possible?

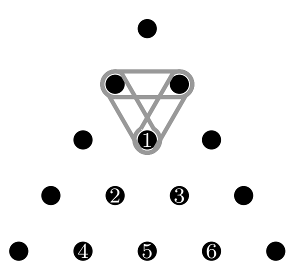
Solution by Florencio Cano Vargas, Inca, Spain.

We classify the equilateral triangles according to the length of its sides ℓ.

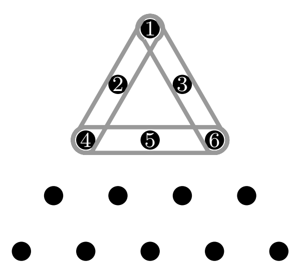
If $\ell = 1$ then we have 16 equilateral triangles, 10 of type 1a as in the diagram on the left and 6 of type 1b as in the diagram on the right. In each case one triangle of each type is shown, numbered 1, and the other numbers indicate where the corresponding vertex would be in the other triangles. A similar numbering system is used for the other diagrams.



Type 1a



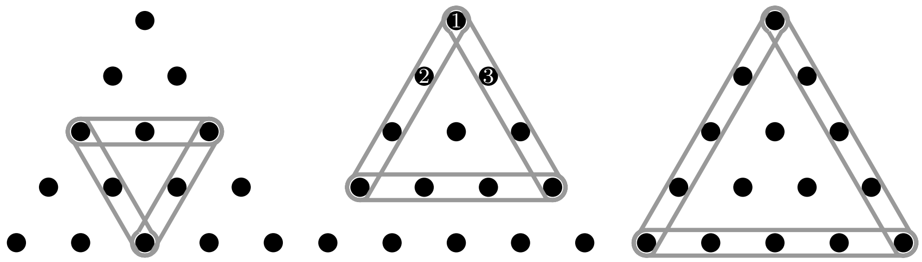
Type 1b



Type 2a

If $\ell = 2$ we have 7 different equilateral triangles, 6 of type 2a and 1 of type 2b. If $\ell = 3$ we have 3 equilateral triangles (type 3). If $\ell = 4$ then we have only 1 equilateral triangle (type 4).

But we may also have equilateral triangles with non-integer side lengths, as shown in the figures below. If $\ell = \sqrt{3}$ then we have 6 different equilateral triangles, 3 of type 5 and 3 more that are the mirror reflections of the triangles of type 5 in the indicated line. If $\ell = \sqrt{7}$ then we have 2 equilateral triangles, one of type 6 and the other the reflection of the triangle in the indicated line.



Type 2b **Type 3** **Type 4**

Type 5 **Type 6**

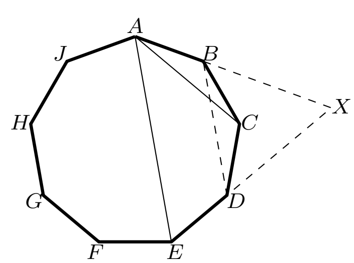
Therefore, in total we have 35 different equilateral triangles.

Also solved by IVAN GERGANOV, student, Kardzhali, Bulgaria; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; JUZ'AN NARI HAIFA, student, SMPN 8, Yogyakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MUHAMMAD LABIB IRFANUDDIN, student, SMP N 8 YOGYAKARTA, Indonesia; MUHAMMAD ROIHAN MUNAJIH, SMA Semesta Bilingual Boarding School, Indonesia; TAUPIEK DIDA PALLEVI, student, SMP N 8 YOGYAKARTA, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KARTIKA CANDRA PUSPITA, student, SMPN 8, Yogyakarta, Indonesia; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; ARIF SETYAWAN, student, SMP N 8 YOGYAKARTA, Indonesia; and STEFANUS RENALDI WIJAYA, student, SMPN 8, Yogyakarta, Indonesia.

M514. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Nonagon $ABCDEFGHI$ is regular. Prove that $AE - AC = AB$.

I. Solution by Juz'an Nari Haifa, student, SMPN 8, Yogyakarta, Indonesia.



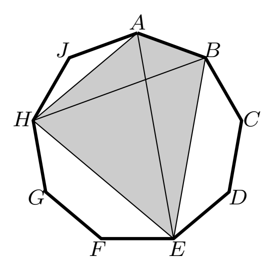
Extending AB and ED and let their point of intersection be X and draw BD. Since $ABCDEFGHI$ is regular, each of its interior angles is $140°$ and each of its exterior angles is $40°$. Clearly ΔBCD is isosceles, and hence $\angle CBD = \angle CDB = \frac{180° - 140°}{2} = 20°$. Hence $\angle XBD = \angle XDB = 60°$ and hence triangles XBD and XAE are equilateral. Since $ABCDEFGHI$ is regular, $AC = BD$ and thus

$$AE = AX = AB + BX = AB + BD = AB + AC$$

and therefore $AE - AC = AB$, as desired.

II. Solution by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania.

From Ptolemy's theorem applied to the cyclic quadrilateral $ABEH$ we get



$$AE \cdot BH = AH \cdot BE + AB \cdot EH\,.$$

Since $ABCDEFGHI$ is regular, $BH = BE = EH$ and $AH = AC$, so

$$AE \cdot BE = AC \cdot BE + AB \cdot BE\,.$$

thus

$$(AE - AC) \cdot BE = AB \cdot BE\,.$$

whence $AE - AC = AB$.

III. Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let O and R, respectively, be the circumcentre and circumradius of $ABCDEFGHI$. Since the chords AB, AC and AE, respectively, subtend angles of $40°$, $80°$ and $160°$ at O, then $AB = 2R\sin 20°$, $AC = 2R\sin 40°$, and $AE = 2R\sin 80°$. Hence

$$\begin{aligned} AE - AC &= 2R(\sin 80° - \sin 40°) \\ &= 2R\left(2\cos\frac{80° + 40°}{2}\sin\frac{80° - 40°}{2}\right) \\ &= 2R\sin 20° \\ &= AB \end{aligned}$$

as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3 solutions); FLORENCIO CANO VARGAS, Inca, Spain; IOAN VIOREL CODREANU, Secondary School student, Satulung, Maramureş, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain (2 solutions); BRUNO SALGUEIRO FANEGO, Viveiro, Spain; MIHAÏ STOËNESCU, Bischwiller, France; DANIEL VĂCARU, Piteşti, Romania; and the proposer.

M515. *Proposed by Titu Zvonaru, Comăneşti, Romania.*

Without using calculus, determine the minimum and maximum values of

$$\frac{2x}{x^2+2x+2}$$

where x is a real number.

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Since $x^2+2x+2=(x+1)^2+1>0$ for all values of x, then if x is nonnegative the fraction is nonnegative and if x is nonpositive the fraction is nonpositive. Thus the maximum occurs when $x \geq 0$ and the minimum when $x \leq 0$.

Let $x>0$. Since $x+\frac{2}{x} \geq 2\sqrt{2}$ with equality if and only if $x=\frac{2}{x}$, then we have

$$\frac{2x}{x^2+2x+2} = \frac{2}{x+\frac{2}{x}+2} \leq \frac{2}{2+2\sqrt{2}} = \sqrt{2}-1$$

with equality if and only if $x=\sqrt{2}$, thus $\sqrt{2}-1$ is the maximum.

As for the minimum, say m, we write for $x \leq 0$,

$$\frac{2x}{x^2+2x+2} \geq m \iff \frac{2(-x)}{x^2+2x+2} \leq -m \iff \frac{2}{(-x)-2+\frac{2}{-x}} \leq -m$$

Denoting $-x=t \geq 0$ we get

$$\frac{2}{t-2+\frac{2}{t}} \leq \frac{2}{-2+2\sqrt{2}} = -m \Longrightarrow m=-1-\sqrt{2}$$

and

$$\left.\frac{2t}{t^2-2t+2}\right|_{t=\sqrt{2}} = 1+\sqrt{2}$$

The minimum is therefore $-1-\sqrt{2}$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ADRIENNA BINGHAM, Angelo State University, San Angelo, TX, USA; FLORENCIO CANO VARGAS, Inca, Spain; MARIUS DAMIAN, Brăila, Romania and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CORNELIU MĂNESCU-AVRAM, Transportation High School, Ploieşti, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; MIHAÏ STOËNESCU, Bischwiller, France; and the proposer.

M516. *Proposed by Syd Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Show that for any given nonzero integer k there exists at least four distinct ordered pairs (x,y) of integers such that

$$\frac{y^2-1}{x^2-1} = k^2-1.$$

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

If (x, y) is an ordered pair of integers such that $\dfrac{y^2-1}{x^2-1} = k^2 - 1$, then $(-x, y), (x, -y)$, and $(-x, -y)$ are ordered pairs which also satisfy the equation. These four ordered pairs are all distinct if x and y are nonzero. Hence it suffices to find an ordered pair of *positive* integers which satisfy the equation. Moreover, since $(-k)^2 = k^2$, we can suppose, without lose of generality, that k is a nonzero positive integer.

If $k = 1$, then $(x, y) = (2, 1)$ satisfies the equation. If $k = 2$ then $(x, y) = (3, 5)$ is a solution.

If $k \geq 3$ then $(x, y) = (k-1, k^2-k-1)$ gives us:

$$\frac{y^2-1}{x^2-1} = \frac{(y+1)(y-1)}{(x+1)(x-1)} = \frac{k(k-1)(k+1)(k-2)}{k(k-2)} = k^2 - 1\,.$$

Since $k-1$ and k^2-k-1 are both positive, then $(x, y) = (k-1, k^2-k-1)$ is an ordered pairs of integers which satisfies the equation for each $k \geq 3$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ADRIENNA BINGHAM, Angelo State University, San Angelo, TX, USA; FLORENCIO CANO VARGAS, Inca, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CORNELIU MĂNESCU-AVRAM, Transportation High School, Ploieşti, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and the proposers.

M517. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all real solutions of the equation

$$3\sqrt{x+y} + 2\sqrt{8-x} + \sqrt{6-y} = 14.$$

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

Let $8 - x = a^2$ and $6 - y = b^2$, where $a \geq 0$ and $b \geq 0$. Then we get $x + y = 14 - a^2 - b^2$. The initial equation becomes

$$3\sqrt{14-a^2-b^2} + 2a + b = 14$$
$$3\sqrt{14-a^2-b^2} = 14 - 2a - b\,.$$

Squaring both sides and rearranging yields

$$9(14 - a^2 - b^2) = 196 + 4a^2 + b^2 - 56a - 26b + 4ab$$

$$13a^2 + (4b - 56)a + 70 + 10b^2 - 28b = 0\,.$$

Solving the equation for a, we obtain

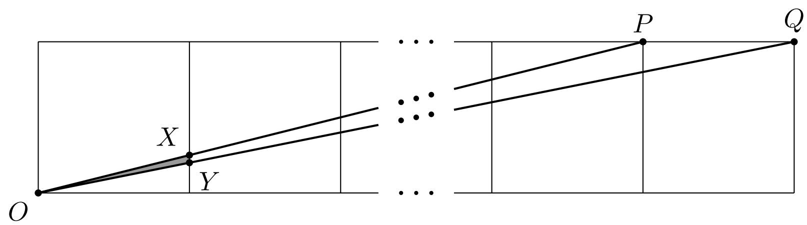
$$\begin{aligned} a &= \frac{28 - 2b \pm \sqrt{126}\sqrt{-b^2+2b-1}}{13} \\ &= \frac{28 - 2b \pm \sqrt{126}\sqrt{-(b-1)^2}}{13}\,. \qquad (1) \end{aligned}$$

If $b-1 \neq 0$ then are no real solutions. Hence, the only real solution occurs when $b=1$. From (1) we get $a=2$. Since $8-x=2^2$ and $6-y=1^2$, the unique solution to the equation is $x=4$ and $y=5$.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; MARIUS DAMIAN, Brăila, Romania and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; JUZ'AN NARI HAIFA, student, SMPN 8, Yogyakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CORNELIU MĂNESCU-AVRAM, Transportation High School, Ploieşti, Romania; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; DANIEL VĂCARU, Piteşti, Romania; and the proposer. One solution with no name was received.

M518. *Selected from a mathematics competition.*

A number of unit squares are placed in a line as shown in the diagram below.



Let O be the bottom left corner of the first square and let P and Q be the top right corners of the 2011^{th} and 2012^{th} squares respectively. When P and Q are connected to O they intersect the right side of the first square at X and Y respectively. Determine the area of triangle OXY.
(*This problem was inspired by question 2c from the 2004 Euclid Contest.*)

Solution by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania.

More generally, replace 2012 by an arbitrary natural number $n \geq 2$ and choose a system of rectangular coordinates with O at the origin, the x-axis containing the bases of the squares and the y-axis along a side of the first square. Consequently, we have $O(0,0), P(n-1,1)$ and $Q(n,1)$. Line OP has equation $(n-1)y=x$ and line OQ has equation $ny=x$, so that $X(1,\frac{1}{n-1})$ and $Y(1,\frac{1}{n})$. The area of the triangle OXY is equal to

$$\frac{y_X-y_Y}{2}=\frac{1}{2n(n-1)}\,.$$

For $n=2012$, this area is equal to $\frac{1}{2\cdot 2011\cdot 2012}$.

Also solved by ADRIENNA BINGHAM, Angelo State University, San Angelo, TX, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FLORENCIO CANO VARGAS, Inca, Spain; ANDHIKA GILANG, student, SMPN 8, Yogyakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MUHAMMAD ROIHAN MUNAJIH, SMA Semesta Bilingual Boarding School, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and DANIEL VĂCARU, Piteşti, Romania.