PROBLEM SOLVER'S TOOLKIT

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The Problem Solver's Toolkit is a new feature in ***Crux Mathematicorum****. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

Harmonic Sets Part 1: Desargues Theorem

The goal of this multi-part essay is to study harmonic sets, but to reach that goal we shall first have to review several basic topics from projective geometry. Since our main interest here is Euclidean geometry, it will be convenient for us to obtain the projective plane from the Euclidean plane by adjoining a *line at infinity*, a line that by definition consists of exactly one new point on every line of the Euclidean plane; two lines of the Euclidean plane share one of these new points if and only if they are parallel. The points and lines of the extended plane form a model of the real projective plane; every pair of lines of the projective extension intersects in exactly one point. In other words we are assuming that the points and lines of our extended plane satisfy two simple properties:

Two points determine a line, on which they both lie;
Two lines determine a point, through which they both pass.

In this way we can speak of parallel lines as intersecting at infinity, thus avoiding annoying special cases in proofs of Euclidean theorems. As a bonus we inherit a duality principle: From any true statement involving points, lines, and incidence we obtain another true statement by mechanically interchanging "point" and "line", and making consequent linguistic changes such as switching "lie" with "pass" and "on" with "through."

We are concerned here with properties of configurations that consist of points and lines. Such properties will be preserved by projecting a configuration from one plane to another. The property that makes it all work was first observed by the seventeenth century French architect Girard Desargues. We say that two triangles ABC and $A'B'C'$ are *perspective from a centre* O if the lines joining corresponding points (namely AA', BB', and CC') are concurrent at O; dually, we say that the triangles are *perspective from an axis* o if the intersection points of corresponding sides (namely AB with $A'B'$ at C'', BC with $B'C'$ at A'', CA with $C'A'$ at B'') are collinear on o.

Theorem (Desargues). Two triangles are perspective from a point if and only if they are perspective from a line.

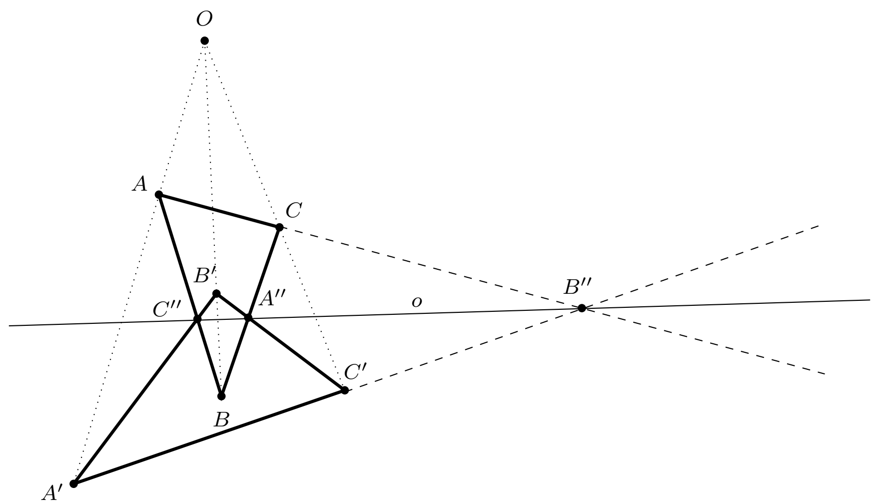

Figure 1: The triangles ABC and $A'B'C'$ are perspective from the point O and from the line o.

There are dozens of proofs of the theorem, some with algebra, some without, that can easily be found elsewhere. Here we will simply observe that the theorem is self dual: the "if" statement is the dual of the "only if" statement. Although we distinguished the centre and axis of the perspectivity in the statement of the theorem, when the ten points and lines of the theorem are considered to be the points and lines of a configuration, then any of the ten points could be chosen to be the centre. The accompanying figure shows the configuration with all ten of its points in the Euclidean plane. The configuration can have as many as four of its points at infinity. Starting with congruent triangles ABC and $A'B'C'$ that have their corresponding sides parallel, we get a Desargues configuration with four points at infinity; a pair of similar triangles with corresponding sides parallel leads to a configuration with three points at infinity.

Exercise 1. Draw a Desargues configuration with exactly one point at infinity. Draw another with exactly two points at infinity.

Exercise 2. Label the ten points of a Desargues configuration by unordered pairs (ij) of the integers from 1 to 5 using the rule that *the three pairs that can be formed from three integers must lie on the same line.* Dually, each line would be labeled by the two integers $[st]$ that do not appear in a label of any of its three points. So, for example, if we attach the pair (12) to A, the three lines through A will be assigned the pairs $[34], [35], [45]$ (in any order), while the remaining points on $[34]$ get the labels (15) and (25). The rest of the labels are now easily determined by observing the rule that the symbols for the three vertices of each of the five triangles formed by points and lines of the configuration must have exactly one integer in common.

The exercise tells us that the automorphism group of a Desargues configuration is the permutation group S_5 of order 5!; in other words, there are 120 ways to label the vertices using the rules of exercise 2.

Exercise 3. How can a person plant ten trees in ten rows of three each?

Exercise 4. Prove that the medians of a triangle are concurrent.

Exercise 5. Two lines are drawn on a sheet of paper, but intersect at a point that is far off the sheet. Given a point on that page, draw the line that joins it to the inaccessible intersection of the given lines.

Exercise 6. Prove that if three triangles are perspective in pairs from the same axis, then their centres of perspective are collinear.

Projective geometry is concerned with collineations of the projective plane; these are the incidence preserving mappings that take points to points and lines to lines. The building blocks of the theory are the *perspective collineations*; these collineations fix all lines through a point (called the *centre*), and all the points on a line (called the *axis*). Familiar examples from Euclidean geometry include the *translations* with both centre and axis at infinity (represented using Cartesian coordinates by $(x, y) \to (x + a, y + b)$), the *dilatations* ($(x, y) \to (ax, ay)$) with centre at the origin and axis at infinity, the *strains* ($(x, y) \to (x, ay)$)) with axis $y = 0$ and centre at infinity, and *shears* $(x, y) \to (x + y, y)$ with axis $y = 0$ and centre at the point of infinity of the axis. In the projective plane, any line o can serve as axis and any point O as centre; the central collineation is then determined by any point A not on o and its image A', which can be any point except O on OA that is not on o. To find the image of any other point B in the plane, we know that if B is on o then B is fixed; otherwise,

(i) the image B' must lie on OB (by the definition of a central collineation), and
(ii) AB must intersect o in a fixed point, call it C'' (because in a projective plane, any two lines must intersect, and every point of o is fixed).

Because collineations preserve incidence, C'' (which is fixed) must lie on both AB and its image line $A'B'$, so that we must have $B' = OB \cap A'C''$. Similarly, for any point C not on OA or OB, if AC meets o at B'' then $C' = OC \cap A'B''$. But how do we know that the image C' of C is well defined by this process? How do we know that if we used B instead of A to define C' we would get the same point? Just look back at Figure 1 and you will see that the desired result (namely, that $BC \cap B'C'$ lies on o) is guaranteed by Desargues's theorem.

Exercise 7. Prove that the product of two perspective collineations that have the same axis is a perspective collineation unless it is the identity.