3819. *Proposed by Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain.*

Let ABC be a triangle with circumcentre O and incentre I. Let ℓ be any line that is perpendicular to OI. Prove that for any point P on ℓ that is inside the triangle, the sum of the distances from P to the sides of ABC is constant.

3820. *Proposed by Michel Bataille, Rouen, France.*

Prove that

$$\frac{2x}{\sinh(2\tanh x)} < (\cosh x)^2 < \frac{2x}{\sinh(2\tanh x)} + x\sinh(2x)$$

for all nonzero real x.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3676. [2011 : 454, 456] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be the sides of a triangle with semiperimeter s, inradius r and circumradius R. Let r' and R' be the inradius and circumradius of a triangle with sides $\sqrt{a(s-a)}$, $\sqrt{b(s-b)}$, and $\sqrt{c(s-c)}$. Prove that

$$Rr' \ \geq\ R'r\,.$$

I. Solution by the proposer.

Let $a' = \sqrt{a(s-a)}$, $b' = \sqrt{b(s-b)}$, and $c' = \sqrt{c(s-c)}$. An easy calculation gives the following useful equality:

$$b'^2 + c'^2 - a'^2 = b(s-b) + c(s-c) - a(s-a) = 2(s-b)(s-c)\,. \qquad (1)$$

From (1), we have $a'^2 - b'^2 - c'^2 < 0 < 2b'c'$, and so consequently, $a' < b' + c'$. In a similar way, $b' < c' + a'$ and $c' < a' + b'$ and so triangles with sides a', b', c' do exist. Let $A'B'C'$ be such a triangle and let A', B', and C' be the angles opposite a', b', and c', respectively. Then using the law of cosines together with (1), the following equality is obtained:

$$\cos A' = \frac{2(s-b)(s-c)}{2\sqrt{bc(s-b)(s-c)}} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sin\frac{A}{2}\,. \qquad (2)$$

The last equality in (2) is because

$$\sin\frac{A}{2} = \frac{r}{\sqrt{r^2+(s-a)^2}} = \frac{rs}{\sqrt{r^2s^2+(s-a)^2s^2}}$$
$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c)+(s-a)^2s^2}} = \sqrt{\frac{(s-b)(s-c)}{bc}}\,. \qquad (3)$$

It is also useful to note that, because $0 < \frac{A}{2} < \frac{\pi}{2}$,

$$\cos\frac{A}{2} = \sqrt{1-\sin^2\frac{A}{2}} = \sqrt{\frac{s(s-a)}{bc}}\,. \qquad (4)$$

Similarly to equation (2), the following are true: $\cos B' = \sin\frac{B}{2}$ and $\cos C' = \sin\frac{C}{2}$. Next, recall the triangle formula:

$$\frac{r}{R} = \cos A + \cos B + \cos C - 1\,. \qquad (5)$$

Combining (2) and (5), yields the following equality:

$$\frac{r'}{R'} = \cos A' + \cos B' + \cos C' - 1 = \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 1\,. \qquad (6)$$

Combining (5) and (6), it follows that we need to prove

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \ge \cos A + \cos B + \cos C\,. \qquad (7)$$

But, this results from the obvious:

$$\left(2\sin\frac{A}{2}\right)\left(1-\cos\frac{B-C}{2}\right) + \left(2\sin\frac{B}{2}\right)\left(1-\cos\frac{C-A}{2}\right)$$
$$+ \left(2\sin\frac{C}{2}\right)\left(1-\cos\frac{A-B}{2}\right) \ge 0\,. \quad (8)$$

Using some trigonometric identities and triangle formulas, it can be shown that (8) can be rewritten as

$$\left(2\sin\frac{A}{2} - \cos B - \cos C\right) + \left(2\sin\frac{B}{2} - \cos C - \cos A\right)$$
$$+ \left(2\sin\frac{C}{2} - \cos A - \cos B\right) \ge 0\,, \quad (9)$$

which is just (7), and the desired inequality is proved. Note that one way to see that (8) and (9) are the same is to reduce them both to the same quantity in terms of just the side lengths a, b, c. This can be done by first applying the cosine of a difference rule, such as $\cos\frac{C-A}{2} = \cos\frac{C}{2}\cos\frac{A}{2} + \sin\frac{C}{2}\sin\frac{A}{2}$, and the double angle formula $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$, then applying equations (3) and (4), and finally substituting $s = \frac{1}{2}(a+b+c)$.

II. Solution by Arkady Alt, San Jose, CA, USA.

Let F be the area of the first triangle with sides a, b, c. Let s' and F' be the semiperimeter and area, respectively, of the second triangle with side lengths $a' = \sqrt{a(s-a)}$, $b' = \sqrt{b(s-b)}$, and $c' = \sqrt{c(s-c)}$. Recall the following formulas for a triangle with sides a, b, c and area F :

$$F = rs \tag{1}$$
$$abc = 4RF \tag{2}$$
$$F = \sqrt{s(s-a)(s-b)(s-c)} \tag{3}$$

This last formula, (3), is referred to as Heron's formula. Note that formulas (1)–(3) can be applied to the second triangle by placing a prime on each of the variables. Using (1) and (2), the goal is to show the following inequality:

$$Rr' \geq R'r \Leftrightarrow \frac{R}{r} \geq \frac{R'}{r'} \Leftrightarrow \frac{4RF}{rsF} \geq \frac{4R'F'}{sr'F'} \Leftrightarrow \frac{abc}{F^2} \geq \frac{a'b'c'}{sr'F'}\,. \tag{4}$$

By observing that

$$2s' = a' + b' + c' = \sqrt{2}\sum_{\text{cyclic}}\sqrt{\frac{a}{2}(s-a)} \leq \frac{\sqrt{2}}{2}\sum_{\text{cyclic}}\left(\frac{a}{2} + (s-a)\right) = \sqrt{2}s$$

and applying (1), the following inequality is established:

$$s \geq \sqrt{2}s' \Leftrightarrow \frac{a'b'c'}{\sqrt{2}s'r'F'} \geq \frac{a'b'c'}{sr'F'} \Leftrightarrow \frac{a'b'c'}{\sqrt{2}F'^2} \geq \frac{a'b'c'}{sr'F'}\,. \tag{5}$$

Because of (5), in order to obtain (4), it suffices to prove the following inequality:

$$\frac{abc}{F^2} \geq \frac{a'b'c'}{\sqrt{2}F'^2} \Leftrightarrow \frac{abc}{F^2} \geq \frac{\sqrt{abc}\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{2}F'^2} \Leftrightarrow \sqrt{2}F'^2\sqrt{sabc} \geq F^3\,, \tag{6}$$

where Heron's formula was used to obtain the second equivalence in (6). Next, using Heron's formula for the triangle with sides a', b', c', and substituting $s' = \frac{1}{2}(a' + b' + c')$, observe that the following equality is true:

$$16F'^2 = 4\sum_{\text{cyclic}} a'^2b'^2 - \left(\sum_{\text{cyclic}} a'^2\right)^2 = 4\sum_{\text{cyclic}} ab(s-a)(s-b) - \left(\sum_{\text{cyclic}} a(s-a)\right)^2. \tag{7}$$

Therefore, from (7) and knowing that $s = \frac{1}{2}(a+b+c)$, the following equality is established:

$$16F'^2 = 4\left((ab+bc+ca)s^2 - s^4 - sabc\right) = 4\left((ab+bc+ca)s^2 - s^4 - 4Rrs^2\right)\,, \tag{8}$$

where (1) and (2) are used to obtain the last equality in (8). Next, using (1)–(3) and $s = \frac{1}{2}(a+b+c)$, it can be shown that $ab+bc+ca = s^2+4Rr+r^2$. Therefore, from (8), the following is obtained:

$$16F'^2 = 4\left((s^2 + 4Rr + r^2)s^2 - s^4 - 4Rrs^2\right) = 4F^2\,. \tag{9}$$

Finally, Euler's formula for the distance between the incenter and circumcenter, combined with formulas (1), (2) and (9), yields the following inequality:

$$\begin{aligned} R \geq 2r \Leftrightarrow 4Rrs^2 \geq 8r^2s^2 & \qquad \Leftrightarrow sabc \geq 8F^2 \\ \Leftrightarrow \sqrt{sabc} \geq 2\sqrt{2}F & \qquad \Leftrightarrow F^2\sqrt{sabc} \geq 2\sqrt{2}F^3 \\ \Leftrightarrow 4F'^2\sqrt{sabc} \geq 2\sqrt{2}F^3 & \qquad \Leftrightarrow \sqrt{2}F'^2\sqrt{sabc} \geq F^3 \,. \end{aligned} \tag{10}$$

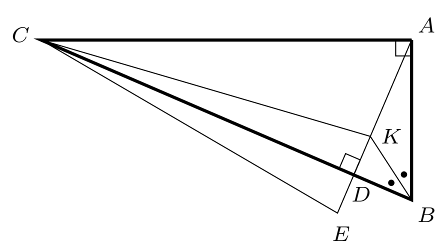
But, this is just (6), and the desired inequality is proved.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MARIAN DINCĂ, Bucharest, Romania; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and TITU ZVONARU, Comăneşti, Romania.

3711. [2012 : 63, 65] *Proposed by Mehmet Sahin, Ankara, Turkey.*

Let ABC be a right triangle with $\angle A = 90°$. Let AD be an altitude, and let the angle bisector of $\angle B$ meet AD in K. If $\angle ACK = 2\angle DCK$ then prove that $KC = 2AD$.

Solution by Dag Jonsson, Uppsala, Sweden.



Since $\triangle ADB$ is similar to $\triangle CDA$ (and similar to $\triangle CAB$),

$$\frac{AD}{AC} = \frac{BD}{BA} \,. \tag{1}$$

Let E be the point on the extension of AD beyond D, such that $DE = KD$. Then the triangles KDC and EDC are congruent, giving $EC = KC$.
Applying the bisector theorem to triangles ABD and ACE, and using equation (1), we obtain

$$2\frac{AD}{AC} = 2\frac{BD}{BA} = 2\frac{KD}{KA} = \frac{KE}{KA} = \frac{EC}{AC} = \frac{KC}{AC}.$$

Thus, $KC = 2AD$.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, *Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; GEORGES MELKI, Fanar, Lebanon; JOEL SCHLOSBERG, Bayside, NY, USA; MIHAÏ STOËNESCU, Bischwiller, France; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.*

3712. [2012 : 63, 65] *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

Prove that for any positive real numbers a, b, c

$$\sqrt{\frac{a(a^2+bc)}{b+c}}+\sqrt{\frac{b(b^2+ca)}{c+a}}+\sqrt{\frac{c(c^2+ab)}{a+b}}\geq a+b+c.$$

I. Composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, NRW, Germany; John G. Heuver, Grande Prairie, AB; and Salem Malikić, student, Simon Fraser University, Burnaby, BC.

By symmetry, we may assume that $a \geq b \geq c > 0$. We prove that

$$\sqrt{\frac{c\,(c^2+ab)}{a+b}}\geq c \tag{1}$$

and that

$$\sqrt{\frac{a\,(a^2+bc)}{b+c}}+\sqrt{\frac{b\,(b^2+ca)}{c+a}}\geq a+b, \tag{2}$$

which together prove the claim.

Inequality (1) is equivalent in succession to,

$$\begin{aligned}\frac{c\left(c^2+ab\right)}{a+b}&\geq c^2,\\ c^2+ab&\geq c\,(a+b)\,,\\ (a-c)\,(b-c)&\geq 0,\end{aligned}$$

which is true by our assumption $a \geq b \geq c > 0$.

Inequality (2) is equivalent to

$$\frac{a\left(a^2+bc\right)}{b+c}+\frac{b\left(b^2+ca\right)}{c+a}+2\sqrt{\frac{a\,(a^2+bc)}{b+c}\cdot\frac{b\,(b^2+ca)}{c+a}}\geq a^2+b^2+2ab. \tag{3}$$

To prove (3), it suffices to prove

$$\frac{a\left(a^2+bc\right)}{b+c}+\frac{b\left(b^2+ca\right)}{c+a}\geq a^2+b^2 \tag{4}$$

and

$$\frac{a\left(a^2+bc\right)}{b+c}\cdot\frac{b\left(b^2+ca\right)}{c+a}\geq a^2b^2. \tag{5}$$

Some algebra shows that (4) is equivalent to $(a-b)^2\left(a^2+ab+b^2-c^2\right)\ge 0$ and that (5) is equivalent to $c\,(a-b)^2\,(a+b)\ge 0$, which are both true by our assumption $a\ge b\ge c>0$; hence the proof is complete.

II. Solution by the proposer, expanded slightly by the editor.

By Hölder's inequality,

$$\left(\sum_{\text{cyclic}}\sqrt{\frac{a\,(a^2+bc)}{b+c}}\right)^2\cdot\sum_{\text{cyclic}}\frac{a^2\,(b+c)}{a^2+bc}\ge\left(\sum_{\text{cyclic}}a\right)^3,$$

so it suffices to show that

$$\sum_{\text{cyclic}}\frac{a^2\,(b+c)}{a^2+bc}\le\sum_{\text{cyclic}}a.$$

We have

$$\begin{aligned}\sum_{\text{cyclic}}\frac{a^2\,(b+c)}{a^2+bc}&=\sum_{\text{cyclic}}\frac{a^2\,(b+c)^2}{(a^2+bc)\,(b+c)}\\&=\sum_{\text{cyclic}}\frac{a^2\,(b+c)^2}{b\,(a^2+c^2)+c\,(a^2+b^2)}.\end{aligned}$$

By the Schwarz inequality in the form

$$\frac{(a_1+a_2)^2}{b_1+b_2}\le\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2},$$

we have

$$\frac{a^2\,(b+c)^2}{b\,(a^2+c^2)+c\,(a^2+b^2)}\le\frac{a^2b^2}{b\,(a^2+c^2)}+\frac{a^2c^2}{c\,(a^2+b^2)}=\frac{a^2b}{a^2+c^2}+\frac{a^2c}{a^2+b^2},$$

so that

$$\begin{aligned}\sum_{\text{cyclic}}\frac{a^2\,(b+c)}{a^2+bc}&\le\sum_{\text{cyclic}}\left(\frac{a^2b}{a^2+c^2}+\frac{a^2c}{a^2+b^2}\right)\\&=\sum_{\text{cyclic}}\left(\frac{a^2b}{a^2+c^2}+\frac{c^2b}{a^2+c^2}\right)\\&=\sum_{\text{cyclic}}b.\end{aligned}$$

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 solutions); MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ÁNGEL PLAZA, *University of Las Palmas de Gran Canaria, Spain; YUNUS TUNCBILEK, Ataturk High School of Science'14, Istanbul, Turkey; and TITU ZVONARU, Comǎneşti, Romania. One incorrect solution was received.*

Several solvers used a generalized Schur inequality, with Tuncbilek explicitly citing a theorem from Pham Kim Hung, Secrets in Inequalities, *Vol 2, GIL Publishing House, 2008. Other solvers used variants of the methods of the two featured solutions. Geupel and Heuver referenced Joe Howard's solution to problem 3554 [2011 : 327] as inspiration for their solutions. Lau used the convexity of the function $1/\sqrt{x}$.*

3713. [2012 : 63, 65] *Proposed by D. M. Bătineţu, Giurgiu, Bucharest and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Compute

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!c_n}}-\frac{n^2}{\sqrt[n]{(2n-1)!!e_n}}\right),$$

where $e_n=\left(1+\dfrac{1}{n}\right)^n$ and $c_n=-\ln n+\displaystyle\sum_{k=1}^{n}\frac{1}{k}$, for any positive integer n.

I. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

The answer is $(e/2)(2-\ln\gamma)$, where $\gamma=\lim_{n\to\infty}c_n=0.577215665\ldots$ is Euler's constant. Recall that $(2n-1)!!=(2n)!2^{-n}(n!)^{-1}=1\cdot3\cdot5\cdots(2n-1)$, the product of the first n odd numbers.

We first establish the following result:

Let $\{a_n\}$ and $\{b_n\}$ be real sequences that satisfy

(a) $\displaystyle\lim_{n\to\infty}\frac{a_n}{n}=\alpha\in(0,\infty)$,

(b) $\displaystyle\lim_{n\to\infty}\frac{b_{n+1}}{a_n}=1$,

(c) $\displaystyle\lim_{n\to\infty}\left(\frac{b_{n+1}}{a_n}\right)^n=\beta\in(0,\infty)$.

Then

$$\lim_{n\to\infty}(b_{n+1}-a_n)=\alpha\ln\beta.$$

Observe that, when $b_{n+1}\neq a_n$,

$$(b_{n+1}-a_n)\left(\frac{a_n}{b_{n+1}-a_n}\right)\ln\left(1+\frac{b_{n+1}-a_n}{a_n}\right)=\frac{a_n}{n}\cdot\ln\left(\frac{b_{n+1}}{a_n}\right)^n.$$

Since $\displaystyle\lim_{n\to\infty}(b_{n+1}-a_n)a_n^{-1}=0$ and $\displaystyle\lim_{t\to0}t^{-1}\ln(1+t)=1$, we deduce that $\displaystyle\lim_{n\to\infty}(b_{n+1}-a_n)=\alpha\ln\beta$. (If $\beta\neq1$, then b_{n+1} is eventually distinct from a_n, while, if $\beta=1$, a slight modification to the argument leads to the same result.)

We now apply this result to

$$b_{n+1}=\sqrt[n+1]{\frac{(n+1)^{2n+2}}{(2n+1)!!c_n}}$$

and

$$a_n = \sqrt[n]{\frac{n^{2n}}{(2n-1)!!e_n}}$$

and show that

$$\lim_{n\to\infty} \frac{a_n}{n} = \frac{e}{2};$$
$$\lim_{n\to\infty} \frac{b_{n+1}}{a_n} = 1;$$

and

$$\lim_{n\to\infty} \left(\frac{b_{n+1}}{a_n}\right)^n = \frac{e^2}{\gamma},$$

from which it will follow that $\lim\limits_{n\to\infty}(b_{n+1} - a_n) = \dfrac{e}{2}(2 - \ln\gamma)$.

First, applying the equal values given by the ratio and root tests, we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{n} &= \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2n-1)!!e_n}} = \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{(2n+1)!!e_{n+1}}\right)\left(\frac{(2n-1)!!e_n}{n^n}\right) \\ &= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(\frac{n+1}{2n+1}\right)\left(\frac{e_n}{e_{n+1}}\right) = \frac{e}{2}.\end{aligned}$$

Secondly,

$$\begin{aligned}\lim_{n\to\infty} \frac{b_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{n}{a_n} \cdot \frac{n+1}{n} \cdot \frac{b_{n+1}}{n+1} = \frac{2}{e} \lim_{n\to\infty} \frac{b_{n+1}}{n+1} \\ &= \frac{2}{e} \lim_{n\to\infty} \sqrt[n+1]{\frac{(n+1)^{n+1}}{(2n+1)!!c_n}} \\ &= \frac{2}{e} \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{(2n+1)!!c_n}\right)\left(\frac{(2n-1)!!c_{n-1}}{n^n}\right) \\ &= \frac{2}{e} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(\frac{n+1}{2n+1}\right)\left(\frac{c_{n-1}}{c_n}\right) = \frac{2}{e}\cdot e \cdot \frac{1}{2} = 1.\end{aligned}$$

Finally,

$$\begin{aligned} n\ln\frac{b_{n+1}}{a_n} &= n\ln\left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!c_n}}\right)\left(\frac{\sqrt[n]{(2n-1)!!e_n}}{n^2}\right) \\ &= 2\ln\left(1+\frac{1}{n}\right)^n + \ln(2n-1)!! + \ln e_n - \frac{n\ln(2n+1)!!}{n+1} - \frac{n\ln c_n}{n+1} \\ &= 3\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + n\left(\frac{1}{n}\sum_{k=1}^{n}\ln(2k-1) - \frac{1}{n+1}\sum_{k=1}^{n}\ln(2k+1)\right) \\ &= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} \\ &\qquad + n\left(\frac{1}{n}\sum_{k=1}^{n-1}\ln(2k+1) - \frac{1}{n+1}\sum_{k=1}^{n-1}\ln(2k+1) - \frac{\ln(2n+1)}{n+1}\right)\end{aligned}$$

$$
\begin{aligned}
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + n\left(\frac{1}{n(n+1)}\sum_{k=1}^{n-1}\ln(2k+1) - \frac{\ln(2n+1)}{n+1}\right)\\
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + \frac{\ln(2n-1)!!}{n+1} - \frac{\ln(2n+1)^n}{n+1}\\
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + \frac{1}{n+1}\ln\frac{(2n-1)!!}{(2n+1)^n}\\
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + \frac{1}{n+1}\ln\left(\frac{(2n)!}{(2n+1)^{2n}}\cdot\frac{(2n+1)^n}{2^n\cdot n!}\right)\\
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + \left(\frac{2n+1}{n+1}\right)\left[\frac{1}{2n+1}\ln\left(\frac{(2n+1)!}{(2n+1)^{2n+1}}\right)\right]\\
&\qquad\qquad + \left(\frac{n}{n+1}\right)\left[\frac{1}{n}\ln\left(\frac{(2n+1)^n}{2^n\cdot n!}\right)\right]
\end{aligned}
$$

We need to apply Stirling's formula:

$$\ln(n!) = n\ln n - n + \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi) + O\left(\frac{1}{n}\right),$$

so that

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{n!}{n^n}\right) = \lim_{n\to\infty}\frac{1}{n}[\ln n! - n\ln n] = -1.$$

Note also that

$$\frac{n^n}{n!} < \frac{(2n+1)^n}{2^n\cdot n!} < \frac{(n+1)^{n+1}}{(n+1)!}.$$

Therefore

$$\lim_{n\to\infty} n\ln\frac{b_{n+1}}{a_n} = 3 - \ln\gamma - 2 + 1 = 2 - \ln\gamma.$$

The solution is complete.

II. Solution by Michel Bataille, Rouen, France.

We shall show that the required limit is $e(1-\ln\sqrt{\gamma})$. We have that

$$
\begin{aligned}
\frac{n^2}{\sqrt[n]{(2n-1)!!e_n}} &= 2n^2\left(1+\frac{1}{n}\right)^{-1}\exp\frac{1}{n}(\ln(n)! - \ln(2n)!)\\
&= 2n^2\left(1-\frac{1}{n}+o(1/n)\right)\\
&\qquad\times\exp\left(\ln n - 1 + \frac{\ln n}{2n} - 2\ln 2 - 2\ln n + 2 - \frac{\ln 2}{2n} - \frac{\ln n}{2n} + o(1/n)\right)\\
&= 2n^2\left(1-\frac{1}{n}+o(1/n)\right)\left(\frac{e}{4n}\right)\exp\left(-\frac{\ln 2}{2n}+o(1/n)\right)\\
&= \frac{en}{2}\left(1-\frac{1}{n}+o(1/n)\right)\left(1-\frac{\ln 2}{2n}+o(1/n)\right)\\
&= \frac{en}{2} - \frac{e}{2}\left(1+\frac{\ln 2}{2}\right) + o(1).
\end{aligned}
$$

Since $c_n = \gamma + o(1)$, we have also that

$$\begin{aligned}
\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!c_n}} &= \exp\left(\frac{-1}{n+1}\ln c_n\right) 2(n+1)^2 \left(\frac{e}{4(n+1)}\right) \exp\left(-\frac{\ln 2}{2(n+1)} + o(1/n)\right) \\
&= \exp\left[\left(\frac{-1}{n} + o(1/n)\right)\ln(\gamma + o(1))\right] \\
&\qquad\qquad \times \left(\frac{e(n+1)}{2}\right)\left(1 - \frac{\ln 2}{2(n+1)} + o(1/n)\right) \\
&= \left(1 - \frac{\ln\gamma}{n} + o(1/n)\right)\left(\frac{en}{2} + \frac{e}{2} - \frac{e\ln 2}{4} + o(1)\right) \\
&= \frac{en}{2} + \frac{e}{2}\left(1 - \frac{\ln 2}{2} - \ln\gamma\right) + o(1).
\end{aligned}$$

The required difference is therefore $e - \frac{e}{2}\ln\gamma + o(1)$, and we obtain the result.

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANASTASIOS KOTRONIS, Athens, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposers.

3714. [2012 : 64, 66] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Given a triangle ABC define points B_1 on side AB and C_1 on AC so that $B_1C_1 \parallel BC$; similarly take C_2 and A_2 on sides BC and BA with $C_2A_2 \parallel CA$, and A_3, B_3 on CA, CB with $A_3B_3 \parallel AB$. Furthermore, denote A'_i, B'_i, C'_i the projections of A_i, B_i, C_i onto the corresponding parallel sides of the given triangle (to form three rectangles such as $B_1B'_1C'_1C_1$).

(a) Prove that if the ratios of the areas of each defined triangle to that of its adjacent rectangle are equal, namely

$$\frac{[AB_1C_1]}{[B_1B'_1C'_1C_1]} = \frac{[BC_2A_2]}{[C_2C'_2A'_2A_2]} = \frac{[CA_3B_3]}{[A_3A'_3B'_3B_3]},$$

then the inradii of those three triangles are also equal.

(b) Determine the ratio of the inradius of the triangle formed by the lines C_1B_1, A_2C_2, B_3A_3 to the inradius of ΔABC.

Composite of solutions by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and by Titu Zvonaru, Comăneşti, Romania.

Let a, b, c be the lengths of the sides of ΔABC, and h be the altitude from A. We set

$$x = \frac{AB_1}{c}, \quad y = \frac{BC_2}{a}, \quad z = \frac{CA_3}{b}.$$

Because triangles are similar if their corresponding sides are parallel, we obtain $B_1C_1 = ax$ and $[AB_1C_1] = x^2[ABC]$. Furthermore, we have $\dfrac{BB_1}{BA} = \dfrac{B_1B'_1}{h}$, which is equivalent to $\dfrac{(1-x)c}{c} = \dfrac{B_1B'_1}{h}$, or

$$BB'_1 = (1-x)h.$$

It follows that

$$[B_1B_1'C_1'C_1] = B_1C_1 \cdot B_1B_1' = ax \cdot (1-x)h = x(1-x) \cdot 2[ABC],$$

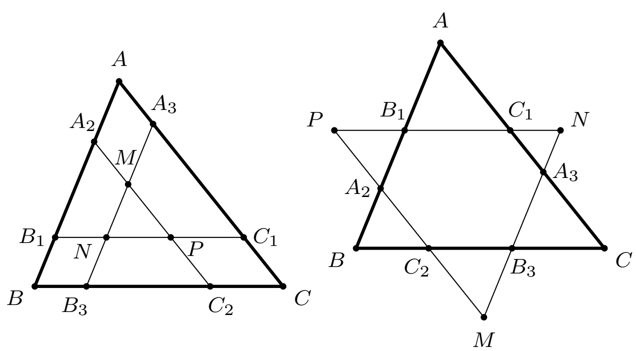
and finally,

$$\frac{[AB_1C_1]}{[B_1B_1'C_1'C_1]} = \frac{x}{2(1-x)}.$$

Solution to (a). Using similar arguments for y and z, we see that the given conditions are equivalent to

$$\frac{x}{1-x} = \frac{y}{1-y} = \frac{z}{1-z};$$

consequently, $x = y = z$, so that the triangles AB_1C_1, BC_2A_2, CA_3B_3 are congruent and, therefore, the inradii of these three triangles are equal; indeed, they are all equal to xr, where r is the inradius of ΔABC.



Solution to (b). The accompanying figures indicate that if we use directed distances, then special cases will not be required. Let MNP be the triangle formed by the lines C_1B_1, A_2C_2, B_3A_3. Using the similar triangles of part (a) we have $C_1A = bx$, $CA_3 = bz$, and, therefore, $A_3C_1 = b(1-x-z)$. From $\Delta AB_1C_1 \sim \Delta A_3NC_1$ we have $\dfrac{A_3C_1}{C_1N} = \dfrac{AC_1}{C_1B_1}$, which is equivalent to $\dfrac{b(1-x-z)}{C_1N} = \dfrac{bx}{ax}$, so that

$$C_1N = a(1-x-z).$$

Similarly,

$$PB_1 = a(1-x-y).$$

It follows that

$$\begin{aligned} PN &= PB_1 + B_1C_1 + C_1N \\ &= a(1-x-y) + ax + a(1-x-z) = a(2-x-y-z). \end{aligned}$$

Because $\Delta MNP \sim \Delta ABC$, we conclude that $|2-x-y-z|r$ is the inradius of ΔMNP, where r is the inradius of ΔABC. More illuminating, the ratio of the dilatation taking the segment BC to NP is $x+y+z-2$, which is positive when the ratios x, y, z are relatively large (as in the figure on the left), and negative when they are relatively small (as on the right).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

In the special case of part (a) when $x = y = z$, the ratio of the inradii of part (b) is $3x - 2$. In terms of the common ratio $\frac{[AB_1C_1]}{[B_1B_1'C_1'C_1]} = \frac{x}{2(1-x)}$, call it t, the ratio of the inradii would be $\frac{2t-2}{2t+1}$.

Swylan observed that the results continue to hold if instead of restricting the new points A_i, B_i, C_i to the sides of the original triangle, we allow them to lie on the extensions of the sides, but only if we use signed areas in part (a). For a counterexample take $x = z = \frac{2}{3}$ and $y = 2$. The resulting area ratios are equal in magnitude (but not in sign), yet one inradius is three times the size of the other two.

3715. [2012 : 64, 66] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let h_a, h_b, h_c be the altitudes, r_a, r_b, r_c the exradii, r the inradius and R the circumradius of a triangle. Prove that

$$\frac{h_a^2}{r_a} + \frac{h_b^2}{r_b} + \frac{h_c^2}{r_c} \geq 4r\left(2 - \frac{r}{R}\right)^2.$$

Solution by Kee-Wai Lau, Hong Kong, China.

Denoting the semiperimeter by $s = \frac{1}{2}(a+b+c)$, we know that $r_a = \frac{rs}{s-a}$ and $h_a = \frac{bc}{2R}$, hence

$$\frac{h_a^2}{r_a} = \frac{b^2c^2(s-a)}{4srR^2}.$$

Similarly,

$$\frac{h_b^2}{r_b} = \frac{c^2a^2(s-b)}{4srR^2} \quad \text{and} \quad \frac{h_c^2}{r_c} = \frac{a^2b^2(s-c)}{4srR^2}.$$

Therefore, the inequality of the problem is equivalent to

$$s\left(a^2b^2 + b^2c^2 + c^2a^2\right) - abc(ab + bc + ca) - 16sr^2(2R - r)^2 \geq 0. \qquad (1)$$

We modify (1) by applying the identities

$$abc = 4srR, \quad ab + bc + ca = s^2 + r^2 + 4rR,$$

and

$$\begin{aligned} a^2b^2 + b^2c^2 + c^2a^2 &= (ab + bc + ca)^2 - 2abc(a + b + c) \\ &= s^4 + r^4 + 16R^2r^2 + 2r^2s^2 + 8Rr^3 - 8s^2Rr \end{aligned}$$

to obtain

$$s^4 + 2r(r - 6R)s^2 - 64r^2R^2 + 68Rr^3 - 15r^4 \geq 0. \qquad (2)$$

The left side of (2) equals $\left(s^2+4Rr-3r^2\right)\left(s^2-16Rr+5r^2\right)$. By Euler's inequality $R \geq 2r$, whence $s^2+4Rr-3r^2>0$. Finally, according to J.C.H. Gerretsen (see Formula 5.8 in O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969), $s^2-16Rr+5r^2 \geq 0$, with equality if and only if the given triangle is equilateral. Thus (2) holds, and we have proved that the original inequality is strict unless the triangle is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; EDMUND SWYLAN, Riga, Latvia; DANIEL VĂCARU, Piteşti, Romania; TITU ZVONARU, Cománeşti, Romania; and the proposer.

3716. [2012 : 64, 66] *Proposed by Michel Bataille, Rouen, France.*

Given positive integers a, u_1, u_2 let u_n be defined by the recursion

$$u_{n+2} = 2au_{n+1} + u_n,\ n \in \mathbb{N}.$$

Show that there exists a positive real number r such that

$$u_{n+2^{k+1}} = \lfloor 1 + r^{2^k} \rfloor \cdot u_{n+2^k} - u_n$$

for all positive integers n, k.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The linear recursion has the characteristic polynomial $x^2-2ax-1$ with the distinct roots $\lambda = a+\sqrt{a^2+1}$ and $-\lambda^{-1}$. Hence, there are real constants b, c such that $u_n = b\lambda^n + c(-\lambda)^{-n}$. For positive integers n, k let the number $q_{n,k}$ be defined by

$$u_{n+2^{k+1}} = (1+q_{n,k}) \cdot u_{n+2^k} - u_n.$$

Then

$$\begin{aligned} q_{n,k} = \frac{u_n - u_{n+2^k} + u_{n+2^{k+1}}}{u_{n+2^k}} &= \frac{\left(\lambda^{2^{k+1}} - \lambda^{2^k} + 1\right)\left(b\lambda^{2n+2^{k+1}} + (-1)^n c\right)}{\lambda^{2^k}\left(b\lambda^{2n+2^{k+1}} + (-1)^n c\right)} \\ &= \lambda^{2^k} - 1 + \lambda^{-2^k}. \end{aligned}$$

Note that $q_{n,k}$ is independent of n. Therefore we can write q_k instead of $q_{n,k}$.

Consider the closed intervals $C_k = \left[q_k^{1/2^{k-1}},\ (q_k+1)^{1/2^{k-1}}\right]$, $\quad k = 1, 2, \ldots$. We have $q_1 = 4a^2+1$ and

$$q_{k+1} = \lambda^{2^{k+1}} - 1 + \lambda^{-2^{k+1}} = \left(\lambda^{2^k} + \lambda^{-2^k}\right)^2 - 3 = (q_k+1)^2 - 3.$$

Hence, q_k is a positive integer for each $k > 0$. We obtain

$$q_k^2 < q_k^2 + 2(q_k - 1) = (q_k+1)^2 - 3 = q_{k+1},$$

that is

$$q_1 < q_2^{1/2} < q_3^{1/4} < q_4^{1/8} < \cdots .$$

Moreover, $q_{k+1} + 1 = (q_k + 1)^2 - 2 < (q_k + 1)^2$. Whence,

$$\cdots < (q_4 + 1)^{1/8} < (q_3 + 1)^{1/4} < (q_2 + 1)^{1/2} < q_1 + 1.$$

Therefore, $C_1 \supset C_2 \supset C_3 \supset C_4 \supset \cdots$. By Cantor's intersection theorem, there is a positive real number r such that

$$r^2 \in \bigcap_{k=1}^{\infty} C_k.$$

Thus, $q_k \leq r^{2^k} \leq q_k + 1$. Here the right inequality is strict because $r^{2^{k+1}} \leq q_{k+1} + 1 < (q_k + 1)^2$ implies $r^{2^k} < q_k + 1$. We conclude

$$q_k = \lfloor r^{2^k} \rfloor.$$

This completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

3717. [2012 : 64, 66] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all real solutions $(x_1, x_2, \ldots, x_n)$ of the system of equations

$$\begin{aligned} x_1 &= \exp\left[\sin\left(x_2 - \sqrt{1 - \ln^2 x_1}\right)\right] \\ x_2 &= \exp\left[\sin\left(x_3 - \sqrt{1 - \ln^2 x_2}\right)\right] \\ &\vdots \\ x_n &= \exp\left[\sin\left(x_1 - \sqrt{1 - \ln^2 x_n}\right)\right]. \end{aligned}$$

Comment by the Editor: There were two submitted solutions in addition to the one given by the proposer. These solutions were quite similar and all claimed that the only solution is $x_1 = x_2 = ... = x_n = 1$. However, they all made the following common mistake: after changing variables in the original equations to some equivalent ones, they took arcsin *of both sides of the resulting equations, ignoring the fact that $y = \sin(x)$ is not equivalent to $x = \arcsin(y)$ unless it can be shown that x is in the interval $[-\pi/2, \pi/2]$ (and it is not in the present case). Actually an in-depth analysis of the implicit function $x = \sin(e^y - \sqrt{1 - x^2})$ shows that it has exactly two fixed points which are both solutions to the given system which therefore must have at least two different solutions. Therefore, this problem remains open.*

3718. [2012 : 65, 67] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b and c be real numbers such that $a > b > c$ and $b + c = 503$.

(i) Find the minimum value of the expression

$$A = \frac{a^2}{a-b} + \frac{b^2}{b-c}.$$

(ii) Determine values of a, b, c for which A attains its minimum value.

I. Solution by Marian Dincă, Bucharest, Romania.

Let $x = a - b$ and $y = b - c$. Using the Arithmetic-Geometric Means Inequality, we obtain that

$$\begin{aligned} A &= \frac{(b+x)^2}{x} + \frac{(c+y)^2}{y} = \left(\frac{b^2}{x} + x\right) + \left(\frac{c^2}{y} + y\right) + 2(b+c) \\ &\geq 2b + 2c + 2(b+c) = 4(b+c) = 4 \cdot 503 = 2012, \end{aligned}$$

with equality if and only if $b = x$ and $c = y$, which is equivalent to $a = 2b = 4c$. This in turn requires that

$$(a, b, c) = \left(\frac{2012}{3}, \frac{1006}{3}, \frac{503}{3}\right).$$

II. Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Salem Malikić, student, Simon Fraser University, Burnaby, BC (independently).

$$\begin{aligned} A &= \left(\frac{a}{\sqrt{a-b}} - 2\sqrt{a-b}\right)^2 + \left(\frac{b}{\sqrt{b-c}} - 2\sqrt{b-c}\right)^2 + 4(b+c) \\ &\geq 4(b+c) = 2012, \end{aligned}$$

with equality if and only if $a = 2(a-b)$ and $b = 2(b-c)$, *i.e.* $(a, b, c) = (2012/3, 1006/3, 503/3)$.

III. Solution by Arkady Alt, San Jose, CA, USA.

Since for any positive reals x, y, and p, $p^2x^2/y \geq 2px - y$ with equality if and only if $px = y$, we have that

$$\begin{aligned} p^2 A &= \frac{p^2a^2}{a-b} + \frac{p^2b^2}{b-c} \\ &\geq 2pa - (a-b) + 2pb - (b-c) = (2p-1)a + 2pb + c. \end{aligned}$$

Taking $p = \frac{1}{2}$ yields that $A \geq 2012$ with equality if and only if $a = 2(a-b)$, $b = 2(b-c)$. With $b + c = 503$ this gives the values of a, b, c recorded above.

IV. Solution by Kee-Wai Lau, Hong Kong, China.

Since $b > c$ and $b + c = 503$, then $2b > 503$. We have that

$$A = \frac{a^2}{a-b} + \frac{b^2}{2b-503} = \frac{(a-2b)^2}{a-b} + \frac{(3b-1006)^2}{2b-503} + 2012.$$

Therefore A attains its minimum value if and only if $3b = 1006$, $a = 2b$.

Also solved by MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; DANIEL VĂCARU, Piteşti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer.

Some solvers resorted to calculus, one using Lagrange Multipliers.

3719. [2012 : 65, 67] *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

Prove that for any positive real numbers a, b, c,

$$\sqrt{\frac{a(a^2+bc)}{b+c}} + \sqrt{\frac{b(b^2+ca)}{c+a}} + \sqrt{\frac{c(c^2+ab)}{a+b}} \geq a+b+c.$$

Composite of essentially the same solution by Radouan Boukharfane, Polytechnique de Montréal, PQ; Oliver Geupel, Brühl, NRW, Germany; John G. Heuver, Grande Prairie, AB; and Itachi Uchiha, Hong Kong, China.

By Hölder's Inequality, we have

$$\begin{aligned}\left(\sum_{\text{cyclic}} \frac{a}{\sqrt{4b^2+bc+4c^2}}\right)^{\frac{2}{3}} &\left(\sum_{\text{cyclic}} a(4b^2+bc+4c^2)\right)^{\frac{1}{3}} \\ &\geq \sum_{\text{cyclic}} \left(\frac{a^{\frac{2}{3}}}{(4b^2+bc+4c^2)^{\frac{1}{3}}}\right)\left(a^{\frac{1}{3}}(4b^2+bc+4c^2)^{\frac{1}{3}}\right) \\ &= a+b+c\end{aligned}$$

so

$$\left(\sum_{\text{cyclic}} \frac{a}{\sqrt{4b^2+bc+4c^2}}\right)^2 \left(\sum_{\text{cyclic}} a(4b^2+bc+4c^2)\right) \geq (a+b+c)^3. \qquad (1)$$

Next, by Schur's Inequality, we have

$$\left(\sum_{\text{cyclic}} a^3\right) + 3abc \geq \sum_{\text{cyclic}} (a^2b+ab^2).$$

Hence,

$$(a+b+c)^3 = \left(\sum_{\text{cyclic}} a^3\right) + 3abc + 3\sum_{\text{cyclic}}(a^2b+ab^2) + 3abc$$
$$\geq 4\sum_{\text{cyclic}} ab(a+b) + 3abc = \sum_{\text{cyclic}} a(4b^2+bc+4c^2)\,. \qquad (2)$$

From (1) and (2) the result follows.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HAOHAO WANG and YANPING XIA, Southeast Missouri State University, Cape Girardeau, MO, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. There was also one incorrect solution. Almost all the submitted solutions used two of the following: convexity and Jensen's Inequality; Cauchy-Schwarz's Inequality; Hölder's Inequality; and Schur's Inequality.

3720. [2012 : 65, 67] *Proposed by Michel Bataille, Rouen, France.*

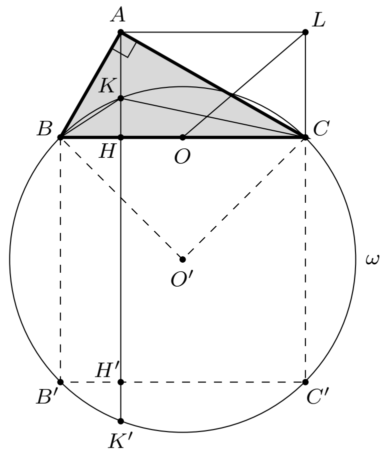
Let ABC be a triangle with $\angle BAC = 90^\circ$, O be the midpoint of BC and H be the foot of the altitude from A. Let K, on segment AH, be such that $\angle BKC = 135^\circ$ and L be such that $AHCL$ is a rectangle. Show that $OL = OB + KH$.

Composite of solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Let $a = BC$ and $x = KH$. Then we have $OB + KH = \frac{a}{2} + x$ and $OL = \sqrt{AH^2 + \left(\frac{a}{2}\right)^2}$. It is sufficient to prove that $OL^2 = (OB+KH)^2$; that is, $AH^2 + \left(\frac{a}{2}\right)^2 = \left(\frac{a}{2}+x\right)^2$, or

$$AH^2 = ax + x^2.$$

Let us denote by ω the circumcircle of triangle BKC and by O' its centre. Because $\angle BKC = 135^\circ$, the angle at O' subtended by the chord BC must be 90°, whence BC is one side of a square $BCC'B'$ inscribed in ω. Denote by K' and H' the points where the line AK meets ω (again) and $B'C'$, respectively. By symmetry, $KH' = HK' = a + x$. Because the chords KK' and BC intersect at H, we have



$$BH \cdot HC = KH \cdot HK' = x \cdot (a + x). \tag{1}$$

Because AH is the altitude to the hypotenuse of the right triangle ABC, we also have

$$BH \cdot HC = AH^2. \tag{2}$$

From (1) and (2) we have $AH^2 = x \cdot (a + x)$, as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; DAG JONSSON, Uppsala, Sweden; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

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