3819. Proposed by Francisco Javier García Capitán, IES Álvaro Cabero, Priego de Córdoba, Spain.

Let $ABC$ be a triangle with circumcentre $O$ and incentre $I$. Let $\ell$ be any line that is perpendicular to $OI$. Prove that for any point $P$ on $\ell$ that is inside the triangle, the sum of the distances from $P$ to the sides of $ABC$ is constant.

3820. Proposed by Michel Bataille, Rouen, France.

Prove that

$$ \frac{2x}{\sinh(2\tanh x)} < (\cosh x)^2 < \frac{2x}{\sinh(2\tanh x)} + x \sinh(2x) $$

for all nonzero real $x$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

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SOLUTIONS

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Let $a$, $b$, and $c$ be the sides of a triangle with semiperimeter $s$, inradius $r$ and circumradius $R$. Let $r'$ and $R'$ be the inradius and circumradius of a triangle with sides $\sqrt{a(s-a)}$, $\sqrt{b(s-b)}$, and $\sqrt{c(s-c)}$. Prove that

$$ Rr' \geq R'r. $$

I. Solution by the proposer.

Let $a' = \sqrt{a(s-a)}$, $b' = \sqrt{b(s-b)}$, and $c' = \sqrt{c(s-c)}$. An easy calculation gives the following useful equality:

$$ b'^2 + c'^2 - a'^2 = b(s-b) + c(s-c) - a(s-a) = 2(s-b)(s-c). \quad (1) $$

From (1), we have $a'^2 - b'^2 + c'^2 < 0 < 2b'c'$, and so consequently, $a' < b' + c'$. In a similar way, $b' < c' + a'$ and $c' < a' + b'$ and so triangles with sides $a'$, $b'$, $c'$ do exist. Let $A'B'C'$ be such a triangle and let $A'$, $B'$, and $C'$ be the angles opposite $a'$, $b'$, and $c'$, respectively. Then using the law of cosines together with (1), the following equality is obtained:

$$ \cos A' = \frac{2(s-b)(s-c)}{2bc(s-b)(s-c)} = \frac{(s-b)(s-c)}{bc} = \sin \frac{A}{2}. \quad (2) $$
The last equality in (2) is because
\[
\sin \frac{A}{2} = \frac{r}{\sqrt{r^2 + (s-a)^2}} = \frac{rs}{\sqrt{r^2 s^2 + (s-a)^2 s^2}} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c) + (s-a)^2 s^2}} = \sqrt{\frac{(s-b)(s-c)}{bc}}.
\]
(3)

It is also useful to note that, because \(0 < \frac{A}{2} < \frac{\pi}{2}\),
\[
\cos \frac{A}{2} = \sqrt{1 - \sin^2 \frac{A}{2}} = \sqrt{\frac{s(s-a)}{bc}}.
\]
(4)

Similarly to equation (2), the following are true: \(\cos B' = \sin \frac{B}{2}\) and \(\cos C' = \sin \frac{C}{2}\).

Next, recall the triangle formula:
\[
\frac{r}{R} = \cos A + \cos B + \cos C - 1.
\]
(5)

Combining (2) and (5), yields the following equality:
\[
\frac{r'}{R'} = \cos A' + \cos B' + \cos C' - 1 = \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} - 1.
\]
(6)

Combining (5) and (6), it follows that we need to prove
\[
\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \geq \cos A + \cos B + \cos C.
\]
(7)

But, this results from the obvious:
\[
\left(2 \sin \frac{A}{2}\right) \left(1 - \cos \frac{B - C}{2}\right) + \left(2 \sin \frac{B}{2}\right) \left(1 - \cos \frac{C - A}{2}\right) + \left(2 \sin \frac{C}{2}\right) \left(1 - \cos \frac{A - B}{2}\right) \geq 0.
\]
(8)

Using some trigonometric identities and triangle formulas, it can be shown that
(8) can be rewritten as
\[
\left(2 \sin \frac{A}{2} - \cos B - \cos C\right) + \left(2 \sin \frac{B}{2} - \cos C - \cos A\right) + \left(2 \sin \frac{C}{2} - \cos A - \cos B\right) \geq 0,
\]
(9)

which is just (7), and the desired inequality is proved. Note that one way to see that (8) and (9) are the same is to reduce them both to the same quantity in terms of just the side lengths \(a, b, c\). This can be done by first applying the cosine of a difference rule, such as \(\cos \frac{C-A}{2} = \cos \frac{C}{2} \cos \frac{A}{2} - \sin \frac{C}{2} \sin \frac{A}{2}\), and the double angle formula \(\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}\), then applying equations (3) and (4), and finally substituting \(s = \frac{1}{2} (a + b + c)\).
II. Solution by Arkady Alt, San Jose, CA, USA.

Let $F$ be the area of the first triangle with sides $a$, $b$, $c$. Let $s'$ and $F'$ be the semiperimeter and area, respectively, of the second triangle with side lengths $a' = \sqrt{a(s-a)}$, $b' = \sqrt{b(s-b)}$, and $c' = \sqrt{c(s-c)}$. Recall the following formulas for a triangle with sides $a$, $b$, $c$ and area $F$:

\[ F = rs \quad (1) \]
\[ abc = 4RF \quad (2) \]
\[ F = \sqrt{s(s-a)(s-b)(s-c)} \quad (3) \]

This last formula, (3), is referred to as Heron’s formula. Note that formulas (1)–(3) can be applied to the second triangle by placing a prime on each of the variables. Using (1) and (2), the goal is to show the following inequality:

\[ \frac{abc}{F^2} \geq \frac{a'b'c'}{\sqrt{2F'^2}} \quad (4) \]

By observing that

\[ 2s' = a' + b' + c' = \sqrt{2} \sum_{\text{cyclic}} \sqrt{\frac{a}{2}(s-a)} \leq \sqrt{2} \sum_{\text{cyclic}} \left( \frac{a}{2} + (s-a) \right) = \sqrt{2}s \]

and applying (1), the following inequality is established:

\[ s \geq \sqrt{2}s' \quad (5) \]

Because of (5), in order to obtain (4), it suffices to prove the following inequality:

\[ \frac{abc}{F^2} \geq \frac{a'b'c'}{\sqrt{2F'^2}} \quad (6) \]

where Heron’s formula was used to obtain the second equivalence in (6). Next, using Heron’s formula for the triangle with sides $a'$, $b'$, $c'$, and substituting $s' = \frac{1}{2}(a' + b' + c')$, observe that the following equality is true:

\[ 16F'^2 = 4 \sum_{\text{cyclic}} a^2b^2 - \left( \sum_{\text{cyclic}} a \right)^2 = 4 \sum_{\text{cyclic}} ab(s-a)(s-b) - \left( \sum_{\text{cyclic}} a(s-a) \right)^2 \quad (7) \]

Therefore, from (7) and knowing that $s = \frac{1}{2}(a + b + c)$, the following equality is established:

\[ 16F'^2 = 4 \left( (ab + bc + ca)s^2 - s^4 - sabc \right) = 4 \left( (ab + bc + ca)s^2 - s^4 - 4Rrs^2 \right) \quad (8) \]

where (1) and (2) are used to obtain the last equality in (8). Next, using (1)–(3) and $s = \frac{1}{2}(a + b + c)$, it can be shown that $ab + bc + ca = s^2 + 4Rr + r^2$. Therefore, from (8), the following is obtained:

\[ 16F'^2 = 4 \left( (s^2 + 4Rr + r^2)s^2 - s^4 - 4Rrs^2 \right) = 4F^2 \quad (9) \]
Finally, Euler’s formula for the distance between the incenter and circumcenter, combined with formulas (1), (2) and (9), yields the following inequality:

\[ R \geq 2r \iff 4Rrs^2 \geq 8r^2s^2 \iff sabc \geq 8F^2 \]

\[ \iff \sqrt{sabc} \geq 2\sqrt{2}F \iff F^2\sqrt{sabc} \geq 2\sqrt{2}F^3 \]

\[ \iff 4F^2\sqrt{sabc} \geq 2\sqrt{2}F^3 \iff \sqrt{2}F^2\sqrt{sabc} \geq F^3. \quad (10) \]

But, this is just (6), and the desired inequality is proved.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MARIAN DINCA, Bucharest, Romania; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and TITU ZVONARU, Comănești, Romania.


Let \( \triangle ABC \) be a right triangle with \( \angle A = 90^\circ \). Let \( AD \) be an altitude, and let the angle bisector of \( \angle B \) meet \( AD \) in \( K \). If \( \angle ACK = 2\angle DCK \) then prove that \( KC = 2AD \).

Solution by Dag Jonsson, Uppsala, Sweden.

Since \( \triangle ADB \) is similar to \( \triangle CDA \) (and similar to \( \triangle CAB \)),

\[ \frac{AD}{AC} = \frac{BD}{BA}. \quad (1) \]

Let \( E \) be the point on the extension of \( AD \) beyond \( D \), such that \( DE = KD \). Then the triangles \( KDC \) and \( EDC \) are congruent, giving \( EC = KC \).

Applying the bisector theorem to triangles \( ABD \) and \( ACE \), and using equation (1), we obtain

\[ 2\frac{AD}{AC} = 2\frac{BD}{BA} = 2\frac{KD}{KA} = \frac{KE}{KA} = \frac{EC}{AC} = \frac{KC}{AC}. \]

Thus, \( KC = 2AD \).

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar,
Propose that for any positive real numbers \(a, b, c\)

\[
\sqrt{\frac{a(a^2 + bc)}{b + c}} + \sqrt{\frac{b(b^2 + ca)}{c + a}} + \sqrt{\frac{c(c^2 + ab)}{a + b}} \geq a + b + c.
\]

I. Composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, NRW, Germany; John G. Heuver, Grande Prairie, AB; and Salem Malikic, student, Simon Fraser University, Burnaby, BC.

By symmetry, we may assume that \(a \geq b \geq c > 0\). We prove that

\[
\sqrt{\frac{c(c^2 + ab)}{a + b}} \geq c
\]

and that

\[
\sqrt{\frac{a(a^2 + bc)}{b + c}} + \sqrt{\frac{b(b^2 + ca)}{c + a}} \geq a + b,
\]

which together prove the claim.

Inequality (1) is equivalent in succession to,

\[
\frac{c(c^2 + ab)}{a + b} \geq c^2, \\
\frac{c^2 + ab}{c + a} \geq c(a + b), \\
(a - c)(b - c) \geq 0,
\]

which is true by our assumption \(a \geq b \geq c > 0\).

Inequality (2) is equivalent to

\[
\frac{a(a^2 + bc)}{b + c} + \frac{b(b^2 + ca)}{c + a} + 2\sqrt{\frac{a(a^2 + bc)}{b + c}} \cdot \frac{b(b^2 + ca)}{c + a} \geq a^2 + b^2 + 2ab. \tag{3}
\]

To prove (3), it suffices to prove

\[
\frac{a(a^2 + bc)}{b + c} + \frac{b(b^2 + ca)}{c + a} \geq a^2 + b^2 \tag{4}
\]

and

\[
\frac{a(a^2 + bc)}{b + c} \cdot \frac{b(b^2 + ca)}{c + a} \geq a^2b^2. \tag{5}
\]
Some algebra shows that (4) is equivalent to \((a - b)^2 (a^2 + ab + b^2 - c^2) \geq 0\) and that (5) is equivalent to \(c (a - b)^2 (a + b) \geq 0\), which are both true by our assumption \(a \geq b \geq c > 0\); hence the proof is complete.

II. Solution by the proposer, expanded slightly by the editor.

By Hölder’s inequality,

\[
\left( \sum_{\text{cyclic}} \sqrt[2]{\frac{a(a^2 + bc)}{b + c}} \right)^2 \cdot \sum_{\text{cyclic}} \frac{a^2 (b + c)}{a^2 + bc} \geq \left( \sum_{\text{cyclic}} a \right)^3,
\]

so it suffices to show that

\[
\sum_{\text{cyclic}} \frac{a^2 (b + c)}{a^2 + bc} \leq \sum_{\text{cyclic}} a.
\]

We have

\[
\sum_{\text{cyclic}} \frac{a^2 (b + c)}{a^2 + bc} = \sum_{\text{cyclic}} \frac{a^2 (b + c)^2}{(a^2 + bc) (b + c)} = \sum_{\text{cyclic}} \frac{a^2 (b + c)^2}{b(a^2 + c^2) + c(a^2 + b^2)}.
\]

By the Schwarz inequality in the form

\[
\frac{(a_1 + a_2)^2}{b_1 + b_2} \leq \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2},
\]

we have

\[
\frac{a^2 (b + c)^2}{b(a^2 + c^2) + c(a^2 + b^2)} \leq \frac{a^2 b^2}{b(a^2 + c^2)} + \frac{a^2 c^2}{c(a^2 + b^2)} = \frac{a^2 b}{a^2 + c^2} + \frac{a^2 c}{a^2 + b^2},
\]

so that

\[
\sum_{\text{cyclic}} \frac{a^2 (b + c)}{a^2 + bc} \leq \sum_{\text{cyclic}} \left( \frac{a^2 b}{a^2 + c^2} + \frac{a^2 c}{a^2 + b^2} \right) = \sum_{\text{cyclic}} \frac{a^2 b}{a^2 + c^2} + \sum_{\text{cyclic}} \frac{a^2 c}{a^2 + b^2} = \sum_{\text{cyclic}} b.
\]

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 solutions); MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ANGEL PLAZA.

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\[ \lim_{n \to \infty} \left( \frac{(n+1)^2}{n^4 \sqrt{(2n+1)!!}} - \frac{n^2}{\sqrt{(2n-1)!!}} \right), \]

where $e_n = \left(1 + \frac{1}{n}\right)^n$ and $c_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$, for any positive integer $n$.

I. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

The answer is $(e/2)(2 - \ln \gamma)$, where $\gamma = \lim_{n \to \infty} c_n = 0.577215665\ldots$ is Euler’s constant. Recall that $(2n-1)!! = (2n)!2^{-n}(n!)^{-1} = 1 \cdot 3 \cdot 5 \cdots (2n-1)$, the product of the first $n$ odd numbers.

We first establish the following result:

Let $\{a_n\}$ and $\{b_n\}$ be real sequences that satisfy

(a) $\lim_{n \to \infty} \frac{a_n}{n} = \alpha \in (0, \infty),$
(b) $\lim_{n \to \infty} \frac{b_{n+1}}{a_n} = 1,$
(c) $\lim_{n \to \infty} \left(\frac{b_{n+1}}{a_n}\right)^n = \beta \in (0, \infty).$

Then

$$\lim_{n \to \infty} (b_{n+1} - a_n) = \alpha \ln \beta.$$ 

Observe that, when $b_{n+1} \neq a_n$,

$$ (b_{n+1} - a_n) \left(\frac{a_n}{b_{n+1} - a_n}\right) \ln \left(1 + \frac{b_{n+1} - a_n}{a_n}\right) = \frac{a_n}{n} \cdot \ln \left(\frac{b_{n+1}}{a_n}\right)^n. $$

Since $\lim_{n \to \infty} (b_{n+1} - a_n)a_n^{-1} = 0$ and $\lim_{t \to 0} t^{-1} \ln(1 + t) = 1$, we deduce that $\lim_{n \to \infty} (b_{n+1} - a_n) = \alpha \ln \beta$. (If $\beta \neq 1$, then $b_{n+1}$ is eventually distinct from $a_n$, while, if $\beta = 1$, a slight modification to the argument leads to the same result.)

We now apply this result to

$$ b_{n+1} = \frac{n+1}{n} \sqrt{(n+1)^{2n+2} \over (2n+1)!! c_n}. $$

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100/ SOLUTIONS

and

\[ a_n = \sqrt[n]{\frac{n^{2n}}{(2n-1)!!e^n}} \]

and show that

\[ \lim_{n \to \infty} \frac{a_n}{n} = \frac{e}{2}; \]
\[ \lim_{n \to \infty} \frac{b_{n+1}}{a_n} = 1; \]

and

\[ \lim_{n \to \infty} \left( \frac{b_{n+1}}{a_n} \right)^n = \frac{e^2}{\gamma}, \]

from which it will follow that \( \lim_{n \to \infty} (b_{n+1} - a_n) = \frac{e}{2} (2 - \ln \gamma) \).

First, applying the equal values given by the ratio and root tests, we obtain

\[ \lim_{n \to \infty} \frac{a_n}{n} = \lim_{n \to \infty} \sqrt[n]{\frac{n^n}{(2n-1)!!e^n}} = \lim_{n \to \infty} \left( \frac{(n+1)^{n+1}}{(2n+1)!!e_{n+1}} \right) \left( \frac{(2n-1)!!e_n}{n^n} \right) \]
\[ = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \left( \frac{n+1}{2n+1} \right) \left( \frac{e_n}{e_{n+1}} \right) = \frac{e}{2}. \]

Secondly,

\[ \lim_{n \to \infty} \frac{b_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n}{n} \cdot \frac{n+1}{n+1} = \frac{2}{e} \lim_{n \to \infty} \frac{b_{n+1}}{n+1} \]
\[ = \frac{2}{e} \lim_{n \to \infty} \sqrt[n]{\frac{n+1}{(2n+1)!!e_{n+1}}} \]
\[ = \frac{2}{e} \lim_{n \to \infty} \left( \frac{n+1}{2n+1} \right) \left( \frac{(2n-1)!!e_{n-1}}{n^n} \right) \]
\[ = \frac{2}{e} \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \left( \frac{n+1}{2n+1} \right) \left( \frac{e_{n-1}}{e_n} \right) = \frac{2}{e} \cdot e \cdot \frac{1}{2} = 1. \]

Finally,

\[ n \ln \frac{b_{n+1}}{a_n} = n \ln \left( \frac{(n+1)^{n+1}}{n \sqrt{(2n+1)!!e_{n+1}}} \right) \left( \frac{\sqrt{(2n-1)!!e_n}}{n^2} \right) \]
\[ = 2 \ln \left( 1 + \frac{1}{n} \right)^n + \ln(2n-1)!! + \ln e_n - \frac{n \ln(2n+1)!! + n \ln c_n}{n+1} - \frac{n \ln c_n}{n+1} \]
\[ = 3 \left( 1 + \frac{1}{n} \right)^n - \frac{n \ln c_n}{n+1} + \frac{1}{n} \sum_{k=1}^{n} \ln(2k-1) - \frac{1}{n+1} \sum_{k=1}^{n} \ln(2k+1) \]
\[ = 3 \ln \left( 1 + \frac{1}{n} \right)^n - \frac{n \ln c_n}{n+1} \]
\[ + n \left( \frac{1}{n} \sum_{k=1}^{n-1} \ln(2k+1) - \frac{1}{n+1} \sum_{k=1}^{n-1} \ln(2k+1) - \frac{\ln(2n+1)}{n+1} \right) \]

\[
= 3 \ln \left( 1 + \frac{1}{n} \right)^n - \frac{n \ln c_n}{n + 1} + \frac{1}{n + 1} \sum_{k=1}^{n-1} \ln(2k+1) - \frac{\ln(2n+1)}{n+1} \\
= 3 \ln \left( 1 + \frac{1}{n} \right)^n - \frac{n \ln c_n}{n + 1} + \frac{\ln(2n-1)!!}{n + 1} - \frac{\ln(2n+1)^n}{n + 1} \\
= 3 \ln \left( 1 + \frac{1}{n} \right)^n - \frac{n \ln c_n}{n + 1} + \frac{1}{n + 1} \ln \left( \frac{(2n-1)!!}{(2n+1)^n} \right) \\
= 3 \ln \left( 1 + \frac{1}{n} \right)^n - \frac{n \ln c_n}{n + 1} + \frac{1}{n + 1} \ln \left( \frac{(2n+1)^n}{(2n-1)!!} \right) \left[ \frac{1}{2n+1} \ln \left( \frac{(2n+1)!}{(2n-1)!!} \right) \right]
\]

We need to apply Stirling’s formula:

\[
\ln(n!) = n \ln n - n + \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi) + O \left( \frac{1}{n} \right),
\]

so that

\[
\lim_{n \to \infty} \frac{1}{n} \ln \left( \frac{n!}{n^n} \right) = \lim_{n \to \infty} \frac{1}{n} \left[ \ln n! - n \ln n \right] = -1.
\]

Note also that

\[
\frac{n^n}{n!} \left( \frac{2n+1)^n}{2^n} \cdot \frac{2^n}{n!} \right) < (n+1)^{n+1}.
\]

Therefore

\[
\lim_{n \to \infty} n \ln \frac{b_{n+1}}{a_n} = 3 - \ln \gamma - 2 + 1 = 2 - \ln \gamma.
\]

The solution is complete.

II. Solution by Michel Bataille, Rouen, France.

We shall show that the required limit is \( e(1 - \ln \sqrt{\gamma}) \). We have that

\[
\frac{n^2}{\sqrt{(2n-1)!!c_n}} = 2n^2 \left( 1 + \frac{1}{n} \right)^{-1} \exp \frac{1}{n} (\ln(n)! - \ln(2n)!) \\
= 2n^2 \left( 1 - \frac{1}{n} + o(1/n) \right) \times \exp \left( \ln n - 1 + \frac{\ln n}{2n} - 2 \ln 2 - 2 \ln n + 2 - \frac{\ln 2}{2n} - \frac{\ln n}{2n} + o(1/n) \right) \\
= 2n^2 \left( 1 - \frac{1}{n} + o(1/n) \right) \left( \frac{e}{4n} \right) \exp \left( -\frac{\ln 2}{2n} + o(1/n) \right) \\
= \frac{e}{2} \left( 1 - \frac{1}{n} + o(1/n) \right) \left( 1 - \frac{\ln 2}{2n} + o(1/n) \right) \\
= \frac{e}{2} - \frac{e}{2} \left( 1 + \ln \frac{2}{2} + o(1) \right).
\]

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Since \( c_n = \gamma + o(1) \), we have also that
\[
\frac{(n+1)^2}{(2n+1)!!c_n} = \exp\left(\frac{-1}{n+1} \ln c_n\right) 2(n+1)^2 \left(\frac{e}{4(n+1)}\right) \exp\left(-\frac{\ln 2}{2(n+1)} + o(1/n)\right)
\]
\[
= \exp\left[\left(\frac{-1}{n} + o(1/n)\right) \ln(\gamma + o(1))\right]
\]
\[
\times \left(\frac{e(n+1)}{2}\right)^2 \left(1 - \frac{\ln 2}{2(n+1)} + o(1/n)\right)
\]
\[
= \left(1 - \frac{\ln \gamma}{n} + o(1/n)\right) \left(\frac{en}{2} + \frac{e}{2} - \frac{eln2}{4} + o(1)\right)
\]
\[
= \frac{en}{2} + \frac{e}{2} \left(1 - \frac{\ln 2}{2} - \ln \gamma\right) + o(1).
\]

The required difference is therefore \( e - \frac{e}{2} \ln \gamma + o(1) \), and we obtain the result.

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANASTASIOS KOTRONIS, Athens, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposers.


Given a triangle \( \triangle ABC \) define points \( B_1 \) on side \( AB \) and \( C_1 \) on \( AC \) so that \( B_1C_1 \parallel BC \); similarly take \( C_2 \) and \( A_2 \) on sides \( BC \) and \( BA \) with \( C_2A_2 \parallel CA \), and \( A_3 \), \( B_3 \) on \( CA \), \( CB \) with \( A_3B_3 \parallel AB \). Furthermore, denote \( A_i, B_i, C_i \) the projections of \( A_i \), \( B_i, C_i \) onto the corresponding parallel sides of the given triangle (to form three rectangles such as \( B_1B'_1C'_1C_1 \)).

(a) Prove that if the ratios of the areas of each defined triangle to that of its adjacent rectangle are equal, namely
\[
\frac{[AB_1C_1]}{[B_1B'_1C'_1C_1]} = \frac{[BC_2A_2]}{[C_2C'_2A'_2A_2]} = \frac{[CA_3B_3]}{[A_3A'_3B'_3B_3]},
\]
then the inradii of those three triangles are also equal.

(b) Determine the ratio of the inradius of the triangle formed by the lines \( C_1B_1, A_2C_2, B_3A_3 \) to the inradius of \( \triangle ABC \).

Composite of solutions by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and by Titu Zvonaru, Comănești, Romania.

Let \( a, b, c \) be the lengths of the sides of \( \triangle ABC \), and \( h \) be the altitude from \( A \). We set
\[
x = \frac{AB_1}{c}, \quad y = \frac{BC_2}{a}, \quad z = \frac{CA_3}{b}.
\]

Because triangles are similar if their corresponding sides are parallel, we obtain \( B_1C_1 = ax \) and \( [AB_1C_1] = x^2[ABC] \). Furthermore, we have \( \frac{BB_1}{BA} = \frac{B_1B'_1}{h} \), which is equivalent to \( \frac{(1-x)c}{c} = \frac{B_1B'_1}{h} \), or
\[
BB'_1 = (1-x)h.
\]
It follows that
\[ [B_1 B'_1 C_1 C'_1] = B_1 C_1 \cdot B_1 B'_1 = ax \cdot (1 - x)h = x(1 - x) \cdot 2[ABC], \]
and finally,
\[ \frac{[AB_1 C_1]}{[B_1 B'_1 C'_1 C_1]} = \frac{x}{2(1 - x)}. \]

**Solution to (a).** Using similar arguments for \( y \) and \( z \), we see that the given conditions are equivalent to
\[ x(1 - x) = y(1 - y) = z(1 - z); \]
consequently, \( x = y = z \), so that the triangles \( AB_1 C_1, BC_2 A_2, CA_3 B_3 \) are congruent and, therefore, the inradii of these three triangles are equal; indeed, they are all equal to \( xr \), where \( r \) is the inradius of \( \Delta ABC \).

**Solution to (b).** The accompanying figures indicate that if we use directed distances, then special cases will not be required. Let \( MNP \) be the triangle formed by the lines \( C_1 B_1, A_2 C_2, B_3 A_3 \). Using the similar triangles of part (a) we have \( C_1 A = bx, CA_3 = bz \), and, therefore, \( A_3 C_1 = b(1 - x - z) \). From \( \Delta AB_1 C_1 \sim \Delta A_3 NC_1 \) we have \( \frac{A_3 C_1}{C_1 N} = \frac{AC_1}{C_1 B_1} \), which is equivalent to \( \frac{b(1 - x - z)}{C_1 N} = \frac{bx}{ax} \) so that
\[ C_1 N = a(1 - x - z). \]

Similarly,
\[ PB_1 = a(1 - x - y). \]

It follows that
\[ PN = PB_1 + B_1 C_1 + C_1 N = a(1 - x - y) + ax + a(1 - x - z) = a(2 - x - y - z). \]

Because \( \Delta MNP \sim \Delta ABC \), we conclude that \( 2 - x - y - z \) \( r \) is the inradius of \( \Delta MNP \), where \( r \) is the inradius of \( \Delta ABC \). More illuminating, the ratio of the dilatation taking the segment \( BC \) to \( NP \) is \( x + y + z - 2 \), which is positive when

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the ratios $x, y, z$ are relatively large (as in the figure on the left), and negative when they are relatively small (as on the right).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

In the special case of part (a) when $x = y = z$, the ratio of the inradii of part (b) is $3x - 2$. In terms of the common ratio $\frac{|AB_1C_1|}{|A_1B_1C_1|} = \frac{x}{x(1-x)}$, call it $t$, the ratio of the inradii would be $2t^2 - 2t + 1$.

Swylan observed that the results continue to hold if instead of restricting the new points $A_i, B_i, C_i$ to the sides of the original triangle, we allow them to lie on the extensions of the sides, but only if we use signed areas in part (a). For a counterexample take $x = z = \frac{2}{3}$ and $y = 2$. The resulting area ratios are equal in magnitude (but not in sign), yet one inradius is three times the size of the other two.


Let $h_a, h_b, h_c$ be the altitudes, $r_a, r_b, r_c$ the exradii, $r$ the inradius and $R$ the circumradius of a triangle. Prove that

$$\frac{h_a^2}{r_a} + \frac{h_b^2}{r_b} + \frac{h_c^2}{r_c} \geq 4r \left(2 - \frac{r}{R}\right)^2.$$

Solution by Kee-Wai Lau, Hong Kong, China.

Denoting the semiperimeter by $s = \frac{1}{2}(a + b + c)$, we know that $r_a = \frac{rs_a}{s-a}$ and $h_a = \frac{bc}{2R}$, hence

$$\frac{h_a^2}{r_a} = \frac{b^2c^2(s-a)}{4srR^2}.$$

Similarly,

$$\frac{h_b^2}{r_b} = \frac{c^2a^2(s-b)}{4srR^2} \quad \text{and} \quad \frac{h_c^2}{r_c} = \frac{a^2b^2(s-c)}{4srR^2}.$$

Therefore, the inequality of the problem is equivalent to

$$s \left(a^2b^2 + b^2c^2 + c^2a^2\right) - abc(ab + bc + ca) - 16sr^2(2R - r)^2 \geq 0. \quad (1)$$

We modify (1) by applying the identities

$$abc = 4srR, \quad ab + bc + ca = s^2 + r^2 + 4rR,$$

and

$$a^2b^2 + b^2c^2 + c^2a^2 = (ab + bc + ca)^2 - 2abc(a + b + c)$$

$$= s^4 + r^4 + 16R^2r^2 + 2r^2s^2 + 8Rr^3 - 8s^2Rr$$

to obtain

$$s^4 + 2r(r - 6R)s^2 - 64r^2R^2 + 68Rr^3 - 15r^4 \geq 0. \quad (2)$$

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The left side of (2) equals \((s^2 + 4Rr + 3r^2) - 3r^2\). By Euler’s inequality \(R \geq 2r\), whence \(s^2 + 4Rr - 3r^2 > 0\). Finally, according to J.C.H. Gerretsen (see Formula 5.8 in O. Bottema et al., Geometric Inequalities, Wolters-Noordhoff Publ., Groningen, 1969), \(s^2 - 16Rr + 5r^2 \geq 0\), with equality if and only if the given triangle is equilateral. Thus (2) holds, and we have proved that the original inequality is strict unless the triangle is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; EDMUND SWYLAN, Riga, Latvia; DANIEL VĂCARU, Pitești, Romania; TITU ZVONARU, Comănești, Romania; and the proposer.


Given positive integers \(a, u_1, u_2\) let \(u_n\) be defined by the recursion

\[ u_{n+2} = 2au_{n+1} + u_n, \quad n \in \mathbb{N}. \]

Show that there exists a positive real number \(r\) such that

\[ u_{n+2^k+1} = \left\lfloor 1 + r^{2^k} \right\rfloor \cdot u_{n+2^k} - u_n \]

for all positive integers \(n, k\).

Solution by Oliver Geupe, Brühl, NRW, Germany.

The linear recursion has the characteristic polynomial \(x^2 - 2ax - 1\) with the distinct roots \(\lambda = a + \sqrt{a^2 + 1}\) and \(-\lambda^{-1}\). Hence, there are real constants \(b, c\) such that \(u_n = b\lambda^n + c(-\lambda)^{-n}\). For positive integers \(n, k\) let the number \(q_{n,k}\) be defined by

\[ u_{n+2^k+1} = (1 + q_{n,k}) \cdot u_{n+2^k} - u_n. \]

Then

\[ q_{n,k} = \frac{u_n - u_{n+2^k} + u_{n+2^k+1}}{u_{n+2^k}} = \frac{\left(\lambda^{2^{k+1}} - \lambda^{2^k} + 1\right)(b\lambda^{2n+2^{k+1}} + (-1)^{n}c)}{\lambda^{2^k} \left(b\lambda^{2n+2^{k+1}} + (-1)^{n}c\right)} = \lambda^{2^k} - 1 + \lambda^{-2^k}. \]

Note that \(q_{n,k}\) is independent of \(n\). Therefore we can write \(q_k\) instead of \(q_{n,k}\).

Consider the closed intervals \(C_k = \left[\frac{1}{4^k}^{2^{k-1}}, (q_k + 1)^{1/2^{k-1}}\right], \quad k = 1, 2, \ldots\)

We have \(q_1 = 4a^2 + 1\) and

\[ q_{k+1} = \lambda^{2^{k+1}} - 1 + \lambda^{-2^{k+1}} = \left(\lambda^{2^k} + \lambda^{-2^k}\right)^2 - 3 = (q_k + 1)^2 - 3. \]

Hence, \(q_k\) is a positive integer for each \(k > 0\). We obtain

\[ q_k^2 < q_k^2 + 2(q_k - 1) = (q_k + 1)^2 - 3 = q_{k+1}. \]

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that is
\[ q_1 < q_2^{1/2} < q_3^{1/4} < q_4^{1/8} < \cdots. \]
Moreover, \( q_{k+1} + 1 = (q_k + 1)^2 - 2 < (q_k + 1)^2. \) Whence,
\[ \cdots < (q_4 + 1)^{1/8} < (q_3 + 1)^{1/4} < (q_2 + 1)^{1/2} < q_1 + 1. \]
Therefore, \( C_1 \supset C_2 \supset C_3 \supset C_4 \supset \cdots. \) By Cantor’s intersection theorem, there is a positive real number \( r \) such that
\[ r^2 \in \bigcap_{k=1}^{\infty} C_k. \]
Thus, \( q_k \leq r^{2^k} \leq q_k + 1. \) Here the right inequality is strict because \( r^{2^{k+1}} \leq q_{k+1} + 1 < (q_k + 1)^2 \) implies \( r^{2^k} < q_k + 1. \) We conclude
\[ q_k = \lfloor r^{2^k} \rfloor. \]
This completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bay-
side, NY, USA; and the proposer.


Find all real solutions \((x_1, x_2, \ldots, x_n)\) of the system of equations
\[
\begin{align*}
  x_1 &= \exp \left[ \sin \left( x_2 - \sqrt{1 - \ln^2 x_1} \right) \right] \\
  x_2 &= \exp \left[ \sin \left( x_3 - \sqrt{1 - \ln^2 x_2} \right) \right] \\
  &\vdots \\
  x_n &= \exp \left[ \sin \left( x_1 - \sqrt{1 - \ln^2 x_n} \right) \right].
\end{align*}
\]

Comment by the Editor: There were two submitted solutions in addition to the one given by the proposer. These solutions were quite similar and all claimed that the only solution is \( x_1 = x_2 = \ldots = x_n = 1. \) However, they all made the following common mistake: after changing variables in the original equations to some equivalent ones, they took arcsin of both sides of the resulting equations, ignoring the fact that \( y = \sin(x) \) is not equivalent to \( x = \arcsin(y) \) unless it can be shown that \( x \) is in the interval \([-\pi/2, \pi/2]\) (and it is not in the present case). Actually an in-depth analysis of the implicit function \( x = \sin(e^y - \sqrt{1 - x^2}) \) shows that it has exactly two fixed points which are both solutions to the given system which therefore must have at least two different solutions. Therefore, this problem remains open.
Let $a, b$ and $c$ be real numbers such that $a > b > c$ and $b + c = 503$.

(i) Find the minimum value of the expression
\[ A = \frac{a^2}{a - b} + \frac{b^2}{b - c}. \]

(ii) Determine values of $a, b, c$ for which $A$ attains its minimum value.

I. Solution by Marian Dinca, Bucharest, Romania.

Let $x = a - b$ and $y = b - c$. Using the Arithmetic-Geometric Means Inequality, we obtain that
\[ A = \frac{(b + x)^2}{x} + \frac{(c + y)^2}{y} = \left( \frac{b^2}{x} + x \right) + \left( \frac{c^2}{y} + y \right) + 2(b + c) \geq 2b + 2c + 2(b + c) = 4(b + c) = 4 \cdot 503 = 2012, \]
with equality if and only if $b = x$ and $c = y$, which is equivalent to $a = 2b = 4c$. This in turn requires that
\[ (a, b, c) = \left( \frac{2012}{3}, \frac{1006}{3}, \frac{503}{3} \right). \]

II. Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Salem Malikić, student, Simon Fraser University, Burnaby, BC (independently).

\[ A = \left( \frac{a}{\sqrt{a - b}} - 2\sqrt{a - b} \right)^2 + \left( \frac{b}{\sqrt{b - c}} - 2\sqrt{b - c} \right)^2 + 4(b + c) \geq 4(b + c) = 2012, \]
with equality if and only if $a = 2(a - b)$ and $b = 2(b - c)$, i.e. $(a, b, c) = (2012/3, 1006/3, 503/3)$.

III. Solution by Arkady Alt, San Jose, CA, USA.

Since for any positive reals $x$, $y$, and $p$, $p^2x^2/y \geq 2px - y$ with equality if and only if $px = y$, we have that
\[ p^2A = \frac{p^2a^2}{a - b} + \frac{p^2b^2}{b - c} \geq 2pa - (a - b) + 2pb - (b - c) = (2p - 1)a + 2pb + c. \]

Taking $p = \frac{1}{2}$ yields that $A \geq 2012$ with equality if and only if $a = 2(a - b)$, $b = 2(b - c)$. With $b + c = 503$ this gives the values of $a, b, c$ recorded above.
IV. Solution by Kee-Wai Lau, Hong Kong, China.

Since \( b > c \) and \( b + c = 503 \), then \( 2b > 503 \). We have that

\[
A = \frac{a^2}{a - b} + \frac{b^2}{2b - 503} = \frac{(a - 2b)^2}{a - b} + \frac{(3b - 1006)^2}{2b - 503} + 2012.
\]

Therefore \( A \) attains its minimum value if and only if \( 3b = 1006, a = 2b \).

Also solved by MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John’s, NL; JOEL SCHLOSBERG, Bayside, NY, USA; DANIEL VĂCĂRU, Pitești, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer.

Some solvers resorted to calculus, one using Lagrange Multipliers.


Prove that for any positive real numbers \( a, b, c \),

\[
\sqrt{a(a^2 + bc)} + \sqrt{b(b^2 + ca)} + \sqrt{c(c^2 + ab)} \geq a + b + c.
\]

Composite of essentially the same solution by Radouan Boukharfane, Polytechnique de Montréal, PQ; Oliver Geupel, Brühl, NRW, Germany; John G. Hewer, Grande Prairie, AB; and Itachi Uchiha, Hong Kong, China.

By Hölder’s Inequality, we have

\[
\left( \sum_{\text{cyclic}} \frac{a}{\sqrt[4]{4b^2 + bc + 4c^2}} \right)^2 \left( \sum_{\text{cyclic}} a(4b^2 + bc + 4c^2) \right)^{\frac{1}{2}} \geq \sum_{\text{cyclic}} \left( \frac{a^2}{4b^2 + bc + 4c^2} \right) \left( a^{\frac{3}{2}}(4b^2 + bc + 4c^2)^{\frac{1}{2}} \right) = a + b + c
\]

so

\[
\left( \sum_{\text{cyclic}} \frac{a}{\sqrt[4]{4b^2 + bc + 4c^2}} \right)^2 \left( \sum_{\text{cyclic}} a(4b^2 + bc + 4c^2) \right) \geq (a + b + c)^3. \quad (1)
\]

Next, by Schur’s Inequality, we have

\[
\left( \sum_{\text{cyclic}} a^3 \right) + 3abc \geq \sum_{\text{cyclic}} (a^2b + ab^2).
\]
Hence,

\[(a + b + c)^3 = \left( \sum_{\text{cyclic}} a^3 \right) + 3abc + 3 \left( \sum_{\text{cyclic}} (a^2b + ab^2) \right) + 3abc \geq 4 \left( \sum_{\text{cyclic}} ab(a + b) \right) + \sum_{\text{cyclic}} a(4b^2 + bc + 4c^2) \cdot (2)\]

From (1) and (2) the result follows.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; DIONNE BAILEY, ELISIE CAMPBELL, and CHARLES R. DIMMINIE, Angelo State University, San Angelo, TX, USA; CAO MINH QUANG, Nguyen Binh Kiem High School, Vinh Long, Vietnam; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HAOHAO WANG and YANPING XIA, Southeast Missouri State University, Cape Girardeau, MO, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was also one incorrect solution. Almost all the submitted solutions used two of the following: convexity and Jensen’s Inequality; Cauchy-Schwarz’s Inequality; Hölder’s Inequality; and Schur’s Inequality.


Let \(ABC\) be a triangle with \(\angle BAC = 90^\circ\), \(O\) be the midpoint of \(BC\) and \(H\) be the foot of the altitude from \(A\). Let \(K\), on segment \(AH\), be such that \(\angle BKC = 135^\circ\) and \(L\) be such that \(AHCL\) is a rectangle. Show that \(OL = OB + KH\).

Composite of solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia.

Let \(a = BC\) and \(x = KH\). Then we have \(OB + KH = \frac{a}{2} + x\) and \(OL = \sqrt{AH^2 + \left(\frac{a}{2}\right)^2}\). It is sufficient to prove that \(OL^2 = (OB + KH)^2\); that is, \(AH^2 + \left(\frac{a}{2}\right)^2 = \left(\frac{a}{2} + x\right)^2\), or

\[AH^2 = ax + x^2.\]

Let us denote by \(\omega\) the circumcircle of triangle \(BKC\) and by \(O'\) its centre. Because \(\angle BKC = 135^\circ\), the angle at \(O'\) subtended by the chord \(BC\) must be 90\(^\circ\), whence \(BC\) is one side of a square \(BCC'B'\) inscribed in \(\omega\). Denote by \(K'\) and \(H'\) the points where the line \(AK\) meets \(\omega\) (again) and \(B'C'\), respectively. By symmetry, \(KH' = HK' = a + x\). Because the chords \(KK'\) and \(BC\) intersect at \(H\), we have
Because $AH$ is the altitude to the hypotenuse of the right triangle $ABC$, we also have

$$BH \cdot HC = AH^2.$$  (2)

From (1) and (2) we have $AH^2 = x \cdot (a + x)$, as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; DAG JONSSON, Uppsala, Sweden; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Pitești, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.