OC119. Soit $ABCD$ un quadrilatère convexe et P le point dintersection des droites AC et BD. Supposons que $AC + AD = BC + BD$. Démontrez que les bissectrices des angles internes de $\angle ACB$, $\angle ADB$ et $\angle APB$ se coupent en un point.

OC120. Soit $S_r(n) = 1^r + 2^r + \cdots + n^r$ où n est un entier positif et r est un nombre rationnel. Le triplet (a, b, c) est dit élégant si a et b sont des nombres rationnels positifs, c est un entier positif et

$$S_a(n) = (S_b(n))^c$$

pour tout entier positif n. Déterminer tous les triplets élégants.

. . . .

OC116. Find all positive integers n which are 300 times the sum of their digits.

OC117. Find the smallest positive integer m such that for all prime numbers $p > 3$,

$$105 \mid 9^{p^2} - 29^p + m\,.$$

OC118. Find all functions $f : (0, \infty) \longrightarrow \mathbb{R}$ satisfying

$$f(x) + f(y) \le \frac{f(x+y)}{2} \quad \text{and} \quad \frac{f(x)}{x} + \frac{f(y)}{y} \ge \frac{f(x+y)}{x+y}\,,$$

for all $x, y \in (0, \infty)$.

OC119. Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB, \angle ADB$ and $\angle APB$ meet at a common point.

OC120. Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where n is a positive integer and r is a rational number. (a, b, c) is called a nice triple if a, b are positive rationals, c is a positive integer and

$$S_a(n) = (S_b(n))^c$$

for all positive integers n. Find all nice triples.

OLYMPIAD SOLUTIONS

OC56. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function so that for all $a, b \in \mathbb{N}$ the expression $af(a) + bf(b) + 2ab$ is a perfect square. Prove that $f(a) = a$ for all $a \in \mathbb{N}$.

(*Originally question 3 from Iran Team Selection Test, Day 4, 2011.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

If p is an odd prime, then setting $a = b = p$ we get that $2p(p + f(p))$ is a perfect square and hence $p \mid p + f(p)$. Thus, for all odd primes p we have

$$p \mid f(p)\,.$$

Now let $p > f(1) - 2f(2)$ be an odd prime. Substituting $a = p$, $b = 1$ and then $a = p$, $b = 2$ into the given relation, we get positive integers c, d so that

$$c^2 = pf(p) + f(1) + 2p\,,$$

and

$$d^2 = pf(p) + 2f(2) + 4p\,.$$

Hence

$$c^2 < c^2 + 2p + 2f(2) - f(1) = d^2\,,$$

and thus $c + 1 \leq d$. Then

$$2c < d^2 - c^2 = 2p + 2f(2) - f(1) \leq 2(p + f(2))\,.$$

As a consequence, we have

$$pf(p) < c^2 < (p + f(2))^2\,.$$

For sufficiently large p, we have $(p + f(2))^2 < 2p^2$. Thus, there exists a K so that for $p > K$ we have $f(p) < 2p$, and since $p \mid f(p)$ we get

$$f(p) = p \text{ for each prime } p > K\,.$$

Let a be any natural number. We are going to prove $f(a) = a$ by contradiction. Consider the cases $f(a) < a$ and $f(a) > a$ in succession.

First, suppose that $f(a) < a$. For each prime $p > K$, setting $b = p$, the number $af(a) + p^2 + 2ap$ is a perfect square that is less than the perfect square $a^2 + p^2 + 2ap = (a + p)^2$. Thus,

$$af(a) + p^2 + 2ap \leq (a + p - 1)^2\,,$$

so that it must be true that

$$2p + 2a - 1 = (a + p)^2 - (a + p - 1)^2 \leq a(a - f(a))\,,$$

which is false for sufficiently large p. This is a contradiction.

Next, suppose $f(a) > a$. For each prime $p > K$, setting $b = p$, the number $af(a) + p^2 + 2ap$ is a perfect square that is greater than the perfect square $a^2 + p^2 + 2ap = (a + p)^2$. Thus,

$$(a + p + 1)^2 \leq af(a) + p^2 + 2ap\,,$$

so that it must be true that

$$2p + 2a + 1 = (a + p + 1)^2 - (a + p)^2 \leq a(f(a) - a)\,,$$

which is false for sufficiently large p. This is a contradiction, which completes the proof.

OC57. Let ABC be a triangle and A', B',C' be the midpoints of BC, CA, AB respectively. Let P and P' be points in a plane such that $PA = P'A'$, $PB = P'B'$, $PC = P'C'$. Prove that all PP' pass through a fixed point.
(*Originally question 2 from Iran Team Selection Test 2011, Day 4.*)

Solved by Michel Bataille, Rouen, France ; Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Cománeşti, Romania. We give Bataille's writeup.

We use complex coordinates with the origin at the circumcentre of ΔABC and, without loss of generality, we assume the circumcircle has radius 1. We will denote the complex coordinate of the point M by the corresponding small letter m. Then, we have

$$a\bar{a} = b\bar{b} = c\bar{c} = 1\,.$$

The hypothesis $PA = P'A'$ writes as

$$(p-a)(\bar{p}-\bar{a}) = \left(p' - \frac{b+c}{2}\right)\left(\bar{p'} - \overline{\frac{b+c}{2}}\right),$$

or

$$2p'(\bar{b}+\bar{c}) + 2\bar{p'}(b+c) - 4p\bar{a} - 4\bar{p}a - b\bar{c} - \bar{b}c = 4|p'|^2 - 4|p|^2 - 2\,.$$

Similarly

$$2p'(\bar{c}+\bar{a}) + 2\bar{p'}(c+a) - 4p\bar{b} - 4\bar{p}b - c\bar{a} - \bar{c}a = 4|p'|^2 - 4|p|^2 - 2\,.$$

Hence, the difference of the last two equations yields

$$2p'(\bar{a}-\bar{b}) + 2\bar{p'}(a-b) + 4p(\bar{a}-\bar{b}) + 4\bar{p}(a-b) - c(\bar{a}-\bar{b}) - \bar{c}(a-b) = 0\,.$$

In an analogous way we get

$$2p'(\bar{b}-\bar{c}) + 2\bar{p'}(b-c) + 4p(\bar{b}-\bar{c}) + 4\bar{p}(b-c) - a(\bar{b}-\bar{c}) - \bar{a}(b-c) = 0\,.$$

By eliminating $\bar{p'}$ from the last two equations, an easy computation yields $p' + 2p = 3w$ where

$$w = \frac{1}{6}\cdot\frac{\bar{a}(b^2-c^2) + \bar{b}(c^2-a^2) + \bar{c}(a^2-b^2)}{\bar{a}(b-c) + \bar{b}(c-a) + \bar{c}(a-b)}\,.$$

It follows that $\overrightarrow{PP'} = 3\overrightarrow{PW}$, which means that PP' passes through W. As the point W depends only on A, B and C, this proves the claim.

OC58. Find the smallest n for which there exists polynomials $f_1(x), f_2(x), \ldots, f_n(x) \in \mathbb{Q}[x]$ such that

$$f_1^2(x) + f_2^2(x) + \cdots + f_n^2(x) = x^2 + 7\,.$$

(*Originally question 3 from British IMO selection 2011, Day 1.*)

Solved by Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA. There were two incorrect solutions.

The solution is $n = 5$.

The example $f_1(x) = x,\ f_2(x) = 2,\ f_3(x) = f_4(x) = f_5(x) = 1$ shows that $n \le 5$.

To prove that this is the smallest, we use the following result from [1, p. 815]:

> *If F is a field with $\mathrm{char}(F) \neq 2$, then $X^2 + d$ can be written as the sum of n squares in $F(X)$ if and only if either -1 or d is the sum of $n-1$ squares in F.*

As -1 cannot be the sum of squares in $\mathbb{Q}$ and 7 cannot be the sum of three rational squares, it follows that $n = 4$ cannot work.

References

[1] Olga Taussky, *Sums of squares*, American Mathematical Monthly 77, 805-830, 1970.

OC59. Let n be an odd positive integer such that both $\phi(n)$ and $\phi(n+1)$ are powers of two. Prove $n+1$ is a power of two or $n = 5$.
(*Originally question 2 from Serbian Math Olympiad 2011.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Curtis.

If $p^2 \mid n$ then $p \mid \phi(n)$ which is not possible as n is odd. Similarly, if p is odd, p^2 cannot divide $n+1$. Thus we have

$$n = \prod_{i=1}^{r} p_i \,,$$

and

$$n + 1 = 2^{\beta} \prod_{j=1}^{s} q_j \,,$$

where $p_1, \ldots, p_r$ are pairwise distinct odd primes and $q_1, \ldots, q_s$ are pairwise distinct odd primes.

As $\phi(n) = (p_1 - 1) \cdots (p_r - 1)$ is a power of two, each p_i is of the form

$$p_i = 2^{a_i} + 1 \,,$$

for distinct positive integers $a_1, \ldots, a_r$. Similarly, each q_j is of the form

$$q_j = 2^{b_j} + 1 \,,$$

for distinct positive integers $b_1, \ldots, b_s$.

Without loss of generality we have

$$a_1 < a_2 < \cdots < a_r$$

and

$$b_1 < b_2 < \cdots < b_s\,.$$

We also have

$$n = \prod_{i=1}^{r}(2^{a_i}+1)\,,$$

and

$$n+1 = 2^\beta \prod_{j=1}^{s}(2^{b_j}+1)\,,$$

We now use the well known fact that if 2^a+1 is prime then $a = 2^k$ for some k. Thus, there exists k_i, l_j so that $a_i = 2^{k_i}$ and $b_j = 2^{l_j}$. Thus we have

$$1 + \prod_{i=1}^{r}(2^{2^{k_i}}+1) = 2^\beta \prod_{j=1}^{s}(2^{2^{l_j}}+1)\,.$$

It follows that when we expand $\prod_{i=1}^{r}(2^{2^{k_i}}+1)$ and $\prod_{j=1}^{s}(2^{2^{l_j}}+1)$ we get a sum of distinct powers of 2:

$$\prod_{i=1}^{r}(2^{2^{k_i}}+1) = 1 + \sum_{k=1}^{M} 2^{u_k}\,,$$

and

$$\prod_{j=1}^{s}(2^{2^{l_j}}+1) = 1 + \sum_{k=1}^{N} 2^{v_k}\,,$$

where

- $1 \le u_1 < u_2 < \cdots < u_M$;
- $1 \le v_1 < v_2 < \cdots < v_N$;
- $N = 2^s - 1$;
- $M = 2^r - 1$;
- $u_1 = a_1 = 2^{k_1}$ and if $r \ge 2$ then $u_2 = a_2 = 2^{k_2}$;
- $v_1 = b_1 = 2^{l_1}$ and if $s \ge 2$ then $v_2 = b_2 = 2^{l_2}$;

and we have

$$1 + \left(1 + \sum_{k=1}^{M} 2^{u_k}\right) = 2^\beta\left(1 + \sum_{k=1}^{N} 2^{v_k}\right)\,.$$

Let's note that each of M and N is either 0 or odd.

Case 1: $u_1 \geq 2$. In this case we have

$$2+\sum_{k=1}^{M} 2^{u_k} = 2^{\beta}\left(1+\sum_{k=1}^{N} 2^{v_k}\right).$$

As the left hand side is 2 (mod 4) it follows that $\beta = 1$ and

$$\sum_{k=1}^{M} 2^{u_k} = \sum_{k=1}^{N} 2^{v_k+1}.$$

This implies $M = N$ and $u_k = 1 + v_k$ for each $1 \leq k \leq M$.

- If $M = N = 0$, then $r = s = 0$ implying that $n = 1$ and $n+1 = 2$, thus $n+1$ is a power of 2.
- If $M = N = 1$ then $a_1 = u_1 = 1 + v_1 = 1 + b_1$, thus

  $$2^{k_1} = 1 + 2^{l_1}.$$

  This implies $l_1 = 0, k_1 = 1$ and hence $p_1 = 5$ and $q_1 = 3$. Thus in this case we get $n = p_1 = 5$.
- If $M = N \geq 1$ then exactly like in the case $N = M = 1$ we get $p_1 = 5$ and $q_1 = 3$. Moreover, as $a_2 = u_2 = v_2 + 1 = b_2$ we also get $k_2 = 1, l_2 = 1$ which yields $p_2 = 5$, a contradiction.

Case 2: $u_1 = 1$. In this case we have

$$4+\sum_{k=2}^{M} 2^{u_k} = 2^{\beta}\left(1+\sum_{k=1}^{N} 2^{v_k}\right).$$

We know that $u_k \geq k$.

- If $u_k = k$ for all $2 \leq k \leq M$ then we get

  $$2^{M+1} = 4+\sum_{k=2}^{M} 2^{u_k} = 2^{\beta}\left(1+\sum_{k=1}^{N} 2^{v_k}\right).$$

  This implies that $N = 0$ and $\beta = M + 1$, thus

  $$n + 1 = 2^{\beta}.$$
- If $u_k > k$ some $2 \leq k \leq M$ then if k_0 is the smallest such k we have

  $$4+\sum_{k=2}^{M} 2^{u_k} = 2^{k_0} + \sum_{k=k_0+1}^{M} 2^{u_k},$$

and

$$k_0 < u_{k_0+1} < u_{k_0+2} < \cdots < u_M .$$

Thus we get

$$2^{k_0} + \sum_{k=k_0+1}^{M} 2^{u_k} = 2^{\beta} + \sum_{k=1}^{N} 2^{v_k+\beta} .$$

Using the uniqueness of representation of an integer as sum of distinct powers of 2, as

$$k_0 < u_{k_0+1} < u_{k_0+2} < \cdots < u_M$$

and

$$\beta < v_1 + \beta < v_2 + \beta < \cdots < v_N + \beta$$

we get

$$\begin{aligned} k_0 &= \beta \\ u_{k_0+1} &= v_1 + \beta \\ &\vdots \\ u_{M-1} &= v_{N-1} + \beta \\ u_M &= v_N + \beta \end{aligned}$$

From the definition of u_M and v_N we have $u_M = a_1 + \cdots + a_r$ and $v_N = b_1 + \cdots + b_s$. Moreover in this case we have $M \geq 2$ and $u_{M-1} = a_2 + a_3 + \cdots + a_r$ and either $N = s = 1$, in which case $u_{M-1} = \beta$ or $v_{N-1} = b_2 + \cdots + b_s + \beta$. In both situations, subtracting $u_M - u_{M-1}$ yields

$$a_1 = b_1 .$$

As $2^{a_1} + 1 \mid n$ and $2^{b_1} + 1 \mid n + 1$ it follows that $2^{a_1} + 1 \mid \gcd(n, n+1) = 1$ a contradiction.

This completes the proof.

OC60. On a blackboard we write the numbers $1, 2, \ldots, 20$. A move consists of selecting two numbers a, b from the blackboard so that $b \geq a + 2$, erasing a and b and writing instead $a+1$ and $b-1$. Find the maximum number of possible moves. (*Originally question 4 from Moldova Team Selection Test 2011, Day 2.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

The required number is 330.

We prove more generally that the maximum possible number of moves for the numbers $1, 2, \ldots, n$ on the blackboard is

$$\frac{1}{6} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n+1}{2} \right\rceil n. \tag{1}$$

Let $s = (s_1, \ldots, s_n)$ denote a state where s_k copies of the number k are on the blackboard, $s_k \geq 0$. The initial state is $S = (1, 1, \ldots, 1)$. We define the value of state s by $f(s) = \sum_{k=1}^{n} s_k k^2$, which is the sum of the squares of the numbers on the board. If a state s can be moved to a state t by selecting numbers a and b such that $a < b$, then

$$f(s) - f(t) = a^2 + b^2 - (a+1)^2 - (b-1)^2 = 2(b-a-1) \geq 2, \tag{2}$$

with equality if and only if $b = a+2$.

By (2) and $f(t) > 0$, the process of making successive moves does eventually terminate at a final state T, say.

At the final state $T = (t_1, \ldots, t_n)$, the numbers on the blackboard are either identical or they take exactly two distinct neighboring values. Note that the average of the values on the blackboard is always $(n+1)/2$. Therefore, for an even number $n = 2m$, we have $t_m = t_{m+1} = m$ and $t_k = 0$ for $k \notin \{m, m+1\}$. For an odd number $n = 2m+1$, we obtain $t_{m+1} = n$ and $t_k = 0$ for $k \neq m+1$. By (2), the number $(f(S) - f(T))/2$ is an upper bound for the number of moves. For even $n = 2m$, we have

$$\frac{f(S) - f(T)}{2} = \frac{1}{2}\left(\sum_{k=1}^{2m} k^2 - m\left(m^2 + (m+1)^2\right)\right) = \frac{(m-1)(m+1)\cdot 2m}{6},$$

For odd $n = 2m+1$ we similarly obtain

$$\frac{f(S) - f(T)}{2} = \frac{m(m+1)(2m+1)}{6}.$$

This is the number (1) in either case.

We use the notation $s \underset{a}{\longrightarrow} t$ for a move with $b = a+2$. For proving that the bound is attained, it is enough to show that there is a sequence of moves that transforms S to T such that only moves of the form $\underset{a}{\longrightarrow}$ are applied. Here it is:

$$\begin{aligned}
(\overbrace{1,1,\ldots,1}^{n}) \underset{1}{\longrightarrow}\underset{2}{\longrightarrow} \cdots \underset{n-2}{\longrightarrow} & (0,2,\overbrace{1,\ldots,1}^{n-4},2,0) \\
\left(\underset{2}{\longrightarrow}\underset{3}{\longrightarrow} \cdots \underset{n-3}{\longrightarrow}\right)^2 & (0,0,3,\overbrace{1,\ldots,1}^{n-6},3,0,0) \\
\left(\underset{3}{\longrightarrow}\underset{4}{\longrightarrow} \cdots \underset{n-4}{\longrightarrow}\right)^3 & (0,0,0,4,\overbrace{1,\ldots,1}^{n-8},4,0,0,0) \\
\vdots &
\end{aligned}$$

For even numbers $n = 2m$ it ends at

$$(\overbrace{0,\ldots,0}^{m-2}, m-1, 1, 1, m-1, \overbrace{0,\ldots,0}^{m-2}) \left(\underset{m-1}{\longrightarrow}\right)^{m-1} (\overbrace{0,\ldots,0}^{m-1}, m, m, \overbrace{0,\ldots,0}^{m-1}).$$

For odd numbers $n = 2m+1$ it ends at

$$(\overbrace{0,\ldots,0}^{m-1}, m, 1, m, \overbrace{0,\ldots,0}^{m-1}) \left(\underset{m}{\longrightarrow}\right)^{m} (\overbrace{0,\ldots,0}^{m}, n, \overbrace{0,\ldots,0}^{m}).$$

This completes the proof that the bound (1) is attained.