THE CONTEST CORNER

No. 12

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_CCNuméro du problème *(exemple : Tremblay_Julie_CC1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions et réponses aux concours au rédacteur à l'adresse* `crux-contest@smc.math.ca`*. Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er mai 2014** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC56. On considère l'ensemble d'entiers consécutifs $\{1, 2, 3, \ldots, n\}$. On retire de cet ensemble trois entiers qui forment une suite géométrique. Les autres entiers de l'ensemble ont une somme de 6125. Déterminer la plus petite valeur possible de n, ainsi que toutes les suites géométriques correspondantes de trois termes qui ont pu être enlevées.

CC57. On considère un triangle acutangle DEF. On trace le cercle C_1 de diamètre DF et le cercle C_2 de diamètre DE. Les points Y et Z sont situés sur les côtés respectifs DF et DE de manière que les segments EY et FZ soient des hauteurs du triangle DEF. Le segment EY coupe le cercle C_1 en P, tandis que le segment FZ coupe le cercle C_2 en Q. Le prolongement de EY coupe le cercle C_1 en R, tandis que le prolongement de FZ coupe le cercle C_2 en S. Démontrer que les points P, Q, R et S sont cocycliques.

CC58. Déterminer toutes les valeurs réelles de x, y et z pour lesquelles

$$\begin{aligned} x - \sqrt{yz} &= 42 \\ y - \sqrt{zx} &= 6 \\ z - \sqrt{xy} &= -30\,. \end{aligned}$$

CC59. Neuf personnes s'exercent à exécuter une danse triangulaire, une danse qui regroupe trois personnes. À chaque exercice, les neuf personnes se regroupent en trois groupes de trois personnes et chaque groupe s'exerce indépendamment des deux autres. On considère que deux exercices sont différents s'il existe au moins une personne qui n'a pas dansé avec la même paire de personnes dans les deux exercices. Combien peut-il y avoir d'exercices différents ?

CC60. On considère l'équation

$$a_0^2 + a_0a_1 + a_1^2 + a_1a_2 + \cdots + a_{2009}a_{2010} + a_{2010}^2 = 1.$$

Combien de solutions entières cette équation admet-elle ?

. . . .

CC56. From the set of consecutive integers $\{1, 2, 3, \ldots, n\}$, three integers that form a geometric sequence are deleted. The sum of the integers remaining is 6125. Determine the smallest value of n and all three-term geometric sequences that make this possible.

CC57. Triangle DEF is acute. Circle C_1 is drawn with DF as its diameter and circle C_2 is drawn with DE as its diameter. Points Y and Z are on DF and DE respectively so that EY and FZ are altitudes of ΔDEF. EY intersects C_1 at P, and FZ intersects C_2 at Q. EY extended intersects C_1 at R, and FZ extended intersects C_2 at S. Prove that P, Q, R, and S are concyclic points.

CC58. Find all real values of x, y and z such that

$$\begin{aligned} x - \sqrt{yz} &= 42 \\ y - \sqrt{zx} &= 6 \\ z - \sqrt{xy} &= -30\,. \end{aligned}$$

CC59. Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place?

CC60. How many integer solutions are there to

$$a_0^2 + a_0a_1 + a_1^2 + a_1a_2 + \cdots + a_{2009}a_{2010} + a_{2010}^2 = 1?$$

CONTEST CORNER SOLUTIONS

CC6. Determine all pairs of positive integers a and b for which

$$3^{x+a} + 2^{x+a} + 2^x = 2^{x+b} + 3^x$$

is satisfied for some integer x.
(*Inspired by question #6 part b) from the 2012 Euclid contest.*)

Two incomplete solutions were received. Both solvers determined that if $a = 2$ and $b = 5$, then $x = 3$ is a solution to the problem (this corresponds to the problem from the Euclid contest). One of the solvers made no attempt to find other solutions or to show there were none. The other solver made an attempt to show there were no other solutions, but the argument was flawed.

CC7. Let $U = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disc in the plane $\mathbb{R}^2$. A chord of U is naturally defined to be a chord of the unit circle with its distinct endpoints removed. Prove or disprove: there is a bijection $f : \mathbb{R}^2 \to U$ such that every straight line in $\mathbb{R}^2$ is mapped by f onto a chord of U.
(*Originally question #3 from the 2012 Science Atlantic Math Competition(Barry Monson).*)

No solutions were received.

CC8. To see who pays for a pizza, A and B play the following simple game. They shuffle a deck of cards, and then in turns draw cards. The first person to draw an ace pays for the pizza. If A draws first, what is the probability that he buys? (Express your answer as a fraction in lowest terms.)
(*Originally question #6 from the 2012 Science Atlantic Math Competition.*)

Solved by Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Kamesha Strong, student, Auburn University at Montgomery, Montgomery, AL, USA. We give the solution of Strong.

Let $P(A_k)$ be the probability that the first ace drawn is by player A on his k^{th} draw. This occurs when the first $2(k-1)$ cards drawn are not aces and the next card is an ace. Since there are 4 aces in the deck, the first ace must be drawn by the 49th card, so for $k = 1, 2, \ldots, 25$

$$\begin{aligned} P(A_k) &= \frac{{}_{48}P_{2(k-1)}}{{}_{52}P_{2(k-1)}} \cdot \frac{4}{52 - 2(k-1)} \\ &= \frac{4({}_{48}P_{2(k-1)})}{{}_{52}P_{2k-1}} \\ &= \frac{4({}_{52}P_{2(k+1)})}{({}_{52}P_{2k-1})({}_{52}P_4)} \\ &= \frac{4({}_{(51-2k)}P_3)}{{}_{52}P_4}. \end{aligned}$$

And the probability that A buys the pizza is:

$$\begin{aligned} \sum_{k=1}^{25} P(A_k) &= \sum_{k=1}^{25} \frac{4({}_{(51-2k)}P_3)}{{}_{52}P_4} \\ &= \frac{4}{52 \cdot 51 \cdot 50 \cdot 49} \sum_{k=1}^{25} {}_{(2k+1)}P_3 \\ &= \frac{1}{26 \cdot 51 \cdot 25 \cdot 49} \sum_{k=1}^{25} (8k^3 - 2k) \\ &= \frac{1}{26 \cdot 51 \cdot 25 \cdot 49} \left[8 \left(\frac{25 \cdot 26}{2} \right)^2 - 2 \left(\frac{25 \cdot 26}{2} \right) \right] \\ &= \frac{1}{51 \cdot 49} [2 \cdot 25 \cdot 26 - 1] \\ &= \frac{433}{833}. \end{aligned}$$

[*Ed.: Alternately, we can consider the deck consisting of* 4 *cards that are aces, and* 48 *other cards. There* $\binom{52}{4}$ *ways to decide which* 4 *positions the aces are in, each of which is equally likely, so it suffices to determine for which of these configurations* A *will be the first player to draw an ace.*

We consider a bijection from these configurations to themselves formed by swapping the cards in positions $(2i-1, 2i)$ *for* $1 \leq i \leq 26$. *For example, the configuration* $C_1, C_2, C_3, C_4, \ldots, C_{51}, C_{52}$ *would become* $C_2, C_1, C_4, C_3, \ldots, C_{52}, C_{51}$. *Notice that after this bijection, a configuration for which* B *would draw the first ace becomes a configuration for which* A *would draw the first ace. A configuration for which* A *would draw the first ace becomes a configuration for which* B *would draw the first ace unless for some* i, *there was an ace in positions* $2i-1$ *and* $2i$ *and no ace in any positions before* $2i-1$.

We determine the probability that the first two aces occur in positions $2i-1$ *and* $2i$. *For a particular value of* i *there are* $\binom{52-2i}{2}$ *ways that this can occur, so* *the probability is*

$$\begin{aligned}\sum_{i=1}^{25} \frac{\binom{52-2i}{2}}{\binom{52}{4}} &= \sum_{i=1}^{25} \frac{\binom{2i}{2}}{\binom{52}{4}} \\ &= \frac{1}{\binom{52}{4}} \sum_{i=1}^{25} (2i^2 - i) \\ &= \frac{1}{\binom{52}{4}} \left[2\left(\frac{25 \cdot 26 \cdot 51}{6}\right) - \frac{25 \cdot 26}{2} \right] \\ &= \frac{10725}{270725} \\ &= \frac{33}{833}\,.\end{aligned}$$

This tells us that the probability A draws the first ace is

$$\frac{33}{833} + \frac{1}{2}\left(1 - \frac{33}{833}\right) = \frac{433}{833}\,.]$$

CC9. Let $k \geq 3$ be an integer. Let $n = \frac{k(k+1)}{2}$. Let $S \subset \mathbb{Z}_n$ with $\|S\| = k$. Show that $S + S \neq \mathbb{Z}_n$. Note that $\|S\|$ denotes the cardinality of S and $S + S = \{x + y \mid x \in S, y \in S\}$.
(*Originally question #4 from the 2012 University of Waterloo Special K Contest.*)

Solution by Florencio Cano Vargas, Inca, Spain, modified by the editor.

There are n elements in $\mathbb{Z}_n$. If we choose a subset $S \subset \mathbb{Z}_n$ with k elements, $(a_1, a_2, \ldots, a_k)$, then the maximum number of elements in $S + S$ (if all sums are unique), is:

$$\binom{k}{2} + k = \binom{k+1}{2} = \frac{k(k+1)}{2} = n\,.$$

Since $S + S$ and $\mathbb{Z}_n$ could have the same cardinality, to prove that $S + S \neq \mathbb{Z}_n$ it is sufficient to prove that there are "repeated elements" in $S + S$, i.e., there exists two distinct subsets of S, $\{a_i, a_s\} \neq \{a_r, a_j\}$, such that $a_i + a_s = a_r + a_j$.

To prove this let us consider the set

$$S - S = \{\alpha_{ij} = a_i - a_j \mid a_i \in S, a_j \in S, a_i \neq a_j\} \subset \mathbb{Z}_n$$

Since we have imposed that $i \neq j$ then $\alpha_{ij} \neq 0$. There is a total of $k(k-1)$ distinct ordered pairs of elements from S so the cardinality of $S - S$ is at most $k(k-1)$. For $k \geq 3$ this is larger than $n - 1$. Indeed:

$$\begin{aligned}k(k-1) > n - 1 &\Leftrightarrow k(k-1) > \frac{k(k+1)}{2} - 1 \\ &\Leftrightarrow 2k(k-1) > k^2 + k - 2 \\ &\Leftrightarrow k^2 - 3k + 2 > 0\end{aligned}$$

which is always true for $k \geq 3$.

This means that, for $k \geq 3$, by the pigeonhole principle, there exist at least two unique pairs of elements of S whose differences are equal. That is, there will be some elements α_{ij}, α_{rs} with $i \neq j, r \neq s$, and $i \neq r, j \neq s$ such that

$$\begin{aligned} \alpha_{ij} &= \alpha_{rs} \\ a_i - a_j &= a_r - a_s \\ a_i + a_s &= a_r + a_j \,. \end{aligned}$$

So we have proved that there are "repeated elements" in $S+S$, so $\|S+S\| < n = \|\mathbb{Z}_n\|$ and hence $S+S \neq \mathbb{Z}_n$.

CC10. Given a positive integer m, let $d(m)$ be the number of positive divisors of m. Determine all positive integers n such that $d(n) + d(n+1) = 5$.
(*Originally question #2 from the 2012 Sun Life Financial Repêchage Competition.*)

Solved by Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Piteşti, Romania; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania.

We present the solution given by Cano.

Let us consider separately the case $n = 1$. In this case $d(1) + d(2) = 3$ which is not a solution to the problem. Therefore we have to consider $n > 1$.

Let $d'(n)$ represent the number of non-trivial divisors of n, that is, the divisors excluding 1 and n. The condition of the problem can be rewritten as:

$$d'(n) + d'(n+1) = 1\,.$$

If $n = 2k$ for some $k \geq 3$, then since $2 \mid n$ and $k \mid n$, $d'(n) \geq 2$. Since either n or $n+1$ is even, then $n+1 < 6$, and a quick check yields two possible solutions:

$$n = 3, n+1 = 4; d(3) = 2; d(4) = 3 \quad \text{and} \quad n = 4, n+1 = 5; d(4) = 3; d(5) = 2\,.$$

[*Ed.: Alternately, after disposing of the case $n = 1$, note that $d(n) \geq 2$ when $n > 1$, so the only possibilities are n such that $d(n) = 2$ and $d(n+1) = 3$, or $d(n) = 3$ and $d(n+1) = 2$. Note that $d(n) = 2$ if and only if n is prime, and $d(n) = 3$ if and only if $n = p^2$ for some prime p. Since n and $n+1$ are of opposite parity, the only possibilities are $n = 3$, a prime, with $n+1 = 4 = 2^2$; and $n = 4 = 2^2$, with $n+1 = 5$, a prime.*]

If you know of a mathematics contest at the high school or undergraduate level whose problems you would like to see in *Contest Corner*, please send information about the contest to `crux-contest@cms.math.ca`.