CC60. How many integer solutions are there to

$$a_0^2 + a_0a_1 + a_1^2 + a_1a_2 + \cdots + a_{2009}a_{2010} + a_{2010}^2 = 1?$$

CONTEST CORNER SOLUTIONS

CC6. Determine all pairs of positive integers a and b for which

$$3^{x+a} + 2^{x+a} + 2^x = 2^{x+b} + 3^x$$

is satisfied for some integer x.
(*Inspired by question #6 part b) from the 2012 Euclid contest.*)

Two incomplete solutions were received. Both solvers determined that if $a = 2$ and $b = 5$, then $x = 3$ is a solution to the problem (this corresponds to the problem from the Euclid contest). One of the solvers made no attempt to find other solutions or to show there were none. The other solver made an attempt to show there were no other solutions, but the argument was flawed.

CC7. Let $U = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disc in the plane $\mathbb{R}^2$. A chord of U is naturally defined to be a chord of the unit circle with its distinct endpoints removed. Prove or disprove: there is a bijection $f : \mathbb{R}^2 \to U$ such that every straight line in $\mathbb{R}^2$ is mapped by f onto a chord of U.
(*Originally question #3 from the 2012 Science Atlantic Math Competition(Barry Monson).*)

No solutions were received.

CC8. To see who pays for a pizza, A and B play the following simple game. They shuffle a deck of cards, and then in turns draw cards. The first person to draw an ace pays for the pizza. If A draws first, what is the probability that he buys? (Express your answer as a fraction in lowest terms.)
(*Originally question #6 from the 2012 Science Atlantic Math Competition.*)

Solved by Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Kamesha Strong, student, Auburn University at Montgomery, Montgomery, AL, USA. We give the solution of Strong.

Let $P(A_k)$ be the probability that the first ace drawn is by player A on his k^{th} draw. This occurs when the first $2(k-1)$ cards drawn are not aces and the next card is an ace. Since there are 4 aces in the deck, the first ace must be drawn by the 49th card, so for $k = 1, 2, \ldots, 25$

$$\begin{aligned} P(A_k) &= \frac{{}_{48}P_{2(k-1)}}{{}_{52}P_{2(k-1)}} \cdot \frac{4}{52 - 2(k-1)} \\ &= \frac{4({}_{48}P_{2(k-1)})}{{}_{52}P_{2k-1}} \\ &= \frac{4({}_{52}P_{2(k+1)})}{({}_{52}P_{2k-1})({}_{52}P_4)} \\ &= \frac{4({}_{(51-2k)}P_3)}{{}_{52}P_4}. \end{aligned}$$

And the probability that A buys the pizza is:

$$\begin{aligned} \sum_{k=1}^{25} P(A_k) &= \sum_{k=1}^{25} \frac{4({}_{(51-2k)}P_3)}{{}_{52}P_4} \\ &= \frac{4}{52 \cdot 51 \cdot 50 \cdot 49} \sum_{k=1}^{25} {}_{(2k+1)}P_3 \\ &= \frac{1}{26 \cdot 51 \cdot 25 \cdot 49} \sum_{k=1}^{25} (8k^3 - 2k) \\ &= \frac{1}{26 \cdot 51 \cdot 25 \cdot 49} \left[8\left(\frac{25 \cdot 26}{2}\right)^2 - 2\left(\frac{25 \cdot 26}{2}\right) \right] \\ &= \frac{1}{51 \cdot 49} [2 \cdot 25 \cdot 26 - 1] \\ &= \frac{433}{833}. \end{aligned}$$

[*Ed.: Alternately, we can consider the deck consisting of* 4 *cards that are aces, and* 48 *other cards. There* $\binom{52}{4}$ *ways to decide which* 4 *positions the aces are in, each of which is equally likely, so it suffices to determine for which of these configurations* A *will be the first player to draw an ace.*

We consider a bijection from these configurations to themselves formed by swapping the cards in positions $(2i-1, 2i)$ *for* $1 \le i \le 26$. *For example, the configuration* $C_1, C_2, C_3, C_4, \ldots, C_{51}, C_{52}$ *would become* $C_2, C_1, C_4, C_3, \ldots, C_{52}, C_{51}$. *Notice that after this bijection, a configuration for which* B *would draw the first ace becomes a configuration for which* A *would draw the first ace. A configuration for which* A *would draw the first ace becomes a configuration for which* B *would draw the first ace unless for some* i, *there was an ace in positions* $2i-1$ *and* $2i$ *and no ace in any positions before* $2i-1$.

We determine the probability that the first two aces occur in positions $2i-1$ *and* $2i$. *For a particular value of* i *there are* $\binom{52-2i}{2}$ *ways that this can occur, so* *the probability is*

$$\begin{aligned}\sum_{i=1}^{25}\frac{\binom{52-2i}{2}}{\binom{52}{4}} &= \sum_{i=1}^{25}\frac{\binom{2i}{2}}{\binom{52}{4}}\\ &= \frac{1}{\binom{52}{4}}\sum_{i=1}^{25}(2i^2-i)\\ &= \frac{1}{\binom{52}{4}}\left[2\left(\frac{25\cdot 26\cdot 51}{6}\right)-\frac{25\cdot 26}{2}\right]\\ &= \frac{10725}{270725}\\ &= \frac{33}{833}\,.\end{aligned}$$

This tells us that the probability A draws the first ace is

$$\frac{33}{833}+\frac{1}{2}\left(1-\frac{33}{833}\right)=\frac{433}{833}\ .]$$

CC9. Let $k \geq 3$ be an integer. Let $n = \frac{k(k+1)}{2}$. Let $S \subset \mathbb{Z}_n$ with $\|S\| = k$. Show that $S + S \neq \mathbb{Z}_n$. Note that $\|S\|$ denotes the cardinality of S and $S + S = \{x + y \mid x \in S, y \in S\}$.
(*Originally question #4 from the 2012 University of Waterloo Special K Contest.*)

Solution by Florencio Cano Vargas, Inca, Spain, modified by the editor.

There are n elements in $\mathbb{Z}_n$. If we choose a subset $S \subset \mathbb{Z}_n$ with k elements, $(a_1, a_2, \ldots, a_k)$, then the maximum number of elements in $S + S$ (if all sums are unique), is:

$$\binom{k}{2}+k=\binom{k+1}{2}=\frac{k(k+1)}{2}=n\,.$$

Since $S + S$ and $\mathbb{Z}_n$ could have the same cardinality, to prove that $S + S \neq \mathbb{Z}_n$ it is sufficient to prove that there are "repeated elements" in $S + S$, i.e., there exists two distinct subsets of S, $\{a_i, a_s\} \neq \{a_r, a_j\}$, such that $a_i + a_s = a_r + a_j$.

To prove this let us consider the set

$$S - S = \{\alpha_{ij} = a_i - a_j \mid a_i \in S, a_j \in S, a_i \neq a_j\} \subset \mathbb{Z}_n$$

Since we have imposed that $i \neq j$ then $\alpha_{ij} \neq 0$. There is a total of $k(k-1)$ distinct ordered pairs of elements from S so the cardinality of $S - S$ is at most $k(k-1)$. For $k \geq 3$ this is larger than $n - 1$. Indeed:

$$\begin{aligned}k(k-1)>n-1 &\Leftrightarrow k(k-1)>\frac{k(k+1)}{2}-1\\ &\Leftrightarrow 2k(k-1)>k^2+k-2\\ &\Leftrightarrow k^2-3k+2>0\end{aligned}$$

which is always true for $k \geq 3$.

This means that, for $k \geq 3$, by the pigeonhole principle, there exist at least two unique pairs of elements of S whose differences are equal. That is, there will be some elements α_{ij}, α_{rs} with $i \neq j, r \neq s$, and $i \neq r, j \neq s$ such that

$$\begin{aligned} \alpha_{ij} &= \alpha_{rs} \\ a_i - a_j &= a_r - a_s \\ a_i + a_s &= a_r + a_j \,. \end{aligned}$$

So we have proved that there are "repeated elements" in $S+S$, so $\|S+S\| < n = \|\mathbb{Z}_n\|$ and hence $S+S \neq \mathbb{Z}_n$.

CC10. Given a positive integer m, let $d(m)$ be the number of positive divisors of m. Determine all positive integers n such that $d(n) + d(n+1) = 5$.
(*Originally question #2 from the 2012 Sun Life Financial Repêchage Competition.*)

Solved by Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Piteşti, Romania; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Cománeşti, Romania.

We present the solution given by Cano.

Let us consider separately the case $n = 1$. In this case $d(1) + d(2) = 3$ which is not a solution to the problem. Therefore we have to consider $n > 1$.

Let $d'(n)$ represent the number of non-trivial divisors of n, that is, the divisors excluding 1 and n. The condition of the problem can be rewritten as:

$$d'(n) + d'(n+1) = 1\,.$$

If $n = 2k$ for some $k \geq 3$, then since $2 \mid n$ and $k \mid n$, $d'(n) \geq 2$. Since either n or $n+1$ is even, then $n+1 < 6$, and a quick check yields two possible solutions:

$$n = 3, n+1 = 4; d(3) = 2; d(4) = 3 \quad \text{and} \quad n = 4, n+1 = 5; d(4) = 3; d(5) = 2\,.$$

[*Ed.: Alternately, after disposing of the case $n = 1$, note that $d(n) \geq 2$ when $n > 1$, so the only possibilities are n such that $d(n) = 2$ and $d(n+1) = 3$, or $d(n) = 3$ and $d(n+1) = 2$. Note that $d(n) = 2$ if and only if n is prime, and $d(n) = 3$ if and only if $n = p^2$ for some prime p. Since n and $n+1$ are of opposite parity, the only possibilities are $n = 3$, a prime, with $n+1 = 4 = 2^2$; and $n = 4 = 2^2$, with $n+1 = 5$, a prime.*]

If you know of a mathematics contest at the high school or undergraduate level whose problems you would like to see in *Contest Corner*, please send information about the contest to `crux-contest@cms.math.ca`.