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MAYHEM SOLUTIONS

M504. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Inside a right triangle with sides 3, 4, 5, two equal circles are drawn that are tangent to one another and to one leg. One circle of the pair is tangent to the hypotenuse. The other circle of the pair is tangent to the other leg. Determine the radii of the circles in both cases.

[*Ed.: This problem was mistakenly republished as problem 3724 [2012 : 105, 106]. It was noticed and later replaced with a new problem [2012 : 194, 194]. This was not quick enough for a few Crux readers who managed to send in their solutions before the problem was retracted. Several of the submissions generalized the problem much like the featured solution [2012 : 309]. The generalization to an arbitrary triangle is presented below.*]

Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

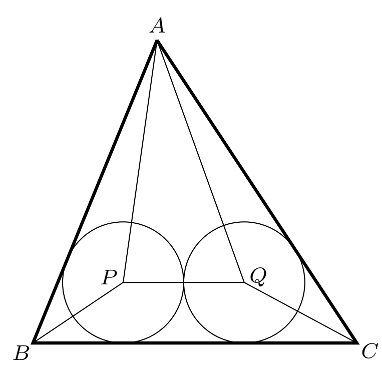
More generally, we will consider an arbitrary triangle with two circles of equal radii tangent to one another and to one of the sides with each of the circles being tangent to one of the remaining two sides. Consider the figure below. Denote the radii of the circles by r, the altitude of the triangle by h, and the area of triangle ABC by $K = ah/2$. Let $BC = a, AC = b$, and $AB = c$. Triangle AQC has area $br/2$, triangle APB has area $cr/2$, trapezoid $BPQC$ has area $(a + 2r)r/2$, and triangle APQ has area $(h - r)r$. Therefore

$$\begin{aligned} K &= br/2 + cr/2 + (a + 2r)r/2 + (h - r)r \\ &= (a/2 + b/2 + c/2 + 2K/a)r, \end{aligned}$$

so

$$r = \frac{2K}{a + b + c + 4K/a},$$

where K can be determined by Heron's formula.



If triangle ABC is a right triangle with hypotenuse c, then $K = ab/2$ and $r = ab/(a + 3b + c)$. For the original problem, taking $a = 4, b = 3$, and $c = 5$ gives $r = 2/3$ and taking $a = 3, b = 4$, and $c = 5$ gives $r = 3/5$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; RADOUAN BOUKHARFANE, Polytechnique de Montréal, PQ; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA*; EDMUND SWYLAN, Riga, Latvia; DANIEL VĂCARU, Piteşti, Romania; ITACHI UCHIHA, Hong Kong, China; and TITU ZVONARU, Comăneşti, Romania*. The asterisk (*) indicates a generalization of the right triangle problem.*

M506. *Proposed by John Grant McLoughlin, University of New Brunswick, Fredericton, NB; and Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

We are trying to create a set of positive integers, that each can be formed using their own digits only, along with any mathematical operations and/or symbols that are familiar to you. Each expression must include at least one symbol/operation; the number of times a digit appears is the same as in the number itself. For example, $1 = \sqrt{1}$, $36 = 6 \times 3!$ and $121 = 11^2$. All valid contributions will be acknowledged.

Ed.: Since the publication of solutions to this problem, the editor has received several pieces of correspondence.

Richard I. Hess, Rancho Palos Verdes, CA, USA, sent the editor expressions for all values from 7 to 105 inclusive, which included some values not represented before such as $10 = .1^{-(0!)}$ *and* $27 = \left\lceil \sqrt{((\sqrt{2+7})!)!} \right\rceil$. *Hess also included more than* 300 *other solutions in the range* 1000 - 2000 *including several formulas good for ranges such as* 1200—$1209 = (0!/.2)!/.1 +$ *units digit. He indicated to the editor by email that he has recently completed* 1100 *more solutions in the range* 2000 - 9999 *which are on the way by regular mail as this is being prepared.*

Stan Wagon, Macalester College, St. Paul, MN, USA, noted that if one used the double factorial function,

$$n!! = \begin{cases} n \times (n-2) \times \cdots \times 5 \times 3 \times 1 & n > 0 \textit{ is odd} \\ n \times (n-2) \times \cdots \times 6 \times 4 \times 2 & n > 0 \textit{ is even} \\ 1 & n = 0 \end{cases}$$

we could get the nicer solution $15 = 1(5!!)$. *Further, Wagon suggests the subfactorial function* $!n$ *which counts the number of derangements of* n *objects (a derangement of* n *objects is a permutation where no object is in its original position). Using this function we get* $1\left\lfloor \sqrt{!6} \right\rfloor = 16$. *He then proceeded to give expressions for* 19, 23, 43 *and* 82 *using various combinations of the ceiling function, square roots and subfactorials.*

As mentioned when the original solutions were published, Hess conjectured that all numbers could be reached using combinations of the factorial, square root, floor and ceiling functions. Wagon noticed this conjecture and wrote some Mathe-

matica programs to find solutions. Wagon quickly reached expression for all numbers from 1 through 100, for example

$$7 = \left\lceil \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\left\lceil \sqrt{7!} \right\rceil !}}}}}}} \right\rceil .$$

Correspondence went on between Hess, Wagon, Al Zimmermann and Joseph DeVincentis. A summary of their findings are given below in an email from Wagon to the editor.

> First, the problem was conjectured by Donald Knuth some years ago using Floor only (and not Ceiling), so the discussion below will use that setup.
>
> Second, Joseph DeVincentis observed that it can be stated in this form: Can one start at 3 and reach any integer by iterating factorial, square root, and floor? This is the form Knuth had as well. This suffices, since any number can be brought down to 3 or 4 by repeating square roots. And getting from 4 to 3 can be done by:
>
> $$\left\lfloor \sqrt{\sqrt{\left\lfloor \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{(4!)!}}}}} \right\rfloor !}} \right\rfloor = 3$$
>
> So if we can get from 3 to n, then we can get from n to n.
>
> So now the computational problem is to start at 3 and start building up a table of what is reachable. Of course, the factorials become unwieldy. Dick Hess programmed his own Stirling formula approximation; Mathematica has LogGamma built-in. So that allows one to go quite far. There are issues of how to store numbers, whether to work with integers or machine reals, whether to store the paths found for every n, and so on. While it seems as if roundoff error would be a killer, in fact the situation is quite stable so that even working with machine reals seems adequate to get reliable paths (which can be checked after the fact using ultra-high precision). So the current state is as follows.
>
> I obtained paths from 3 to every number up to 2000. Dick Hess has gotten up to 40389 by taking 199 generations and keeping track of numbers (i.e., logfactorial numbers, so in fact he is checking up to 48 million factorial) under 48 million. So the first unresolved case is 40390. I have not found much online about whether others have checked this conjecture to these realms. In any case, the evidence is quite strong that all numbers can be reached from 3.
>
> Thanks to Joseph, Al Zimmermann, and Dick Hess for help with this intriguing problem.

The problem has inspired Wagon, as shown in the Macalester College Problem of the Week *problem number* 1171 Four Unary Functions *which can be viewed at*

`http://mathforum.org/wagon/fall13/p1171.html,`

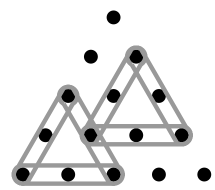
with solutions (large file!) at

`http://mathforum.org/wagon/current_solutions/s1171.html.`

At the time of publication, Wagon has extended the his solutions to 131 110*, inclusive.*

M513. *Proposed by the Mayhem Staff.*

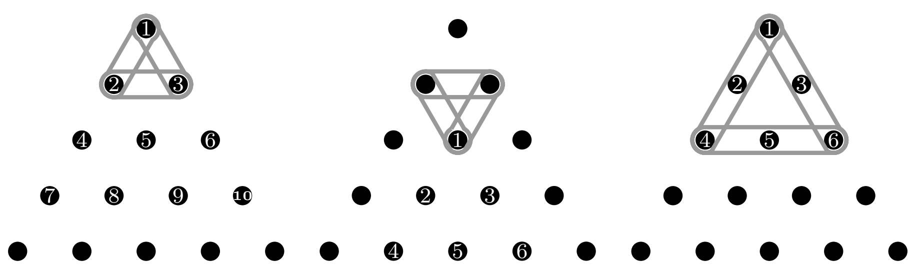
An equilateral triangular grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form equilateral triangles, two different equilateral triangles two centimetres on each side are shown in the diagram. How many different equilateral triangles are possible?



Solution by Florencio Cano Vargas, Inca, Spain.

We classify the equilateral triangles according to the length of its sides ℓ.

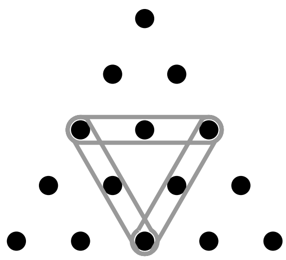
If $\ell = 1$ then we have 16 equilateral triangles, 10 of type 1a as in the diagram on the left and 6 of type 1b as in the diagram on the right. In each case one triangle of each type is shown, numbered 1, and the other numbers indicate where the corresponding vertex would be in the other triangles. A similar numbering system is used for the other diagrams.



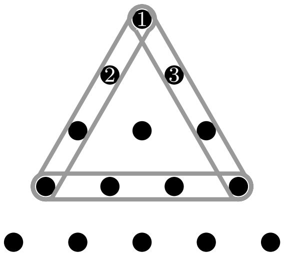
Type 1a **Type 1b** **Type 2a**

If $\ell = 2$ we have 7 different equilateral triangles, 6 of type 2a and 1 of type 2b. If $\ell = 3$ we have 3 equilateral triangles (type 3). If $\ell = 4$ then we have only 1 equilateral triangle (type 4).

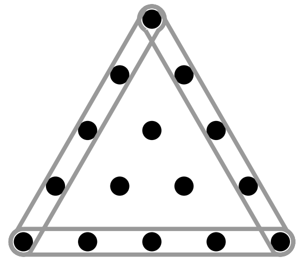
But we may also have equilateral triangles with non-integer side lengths, as shown in the figures below. If $\ell = \sqrt{3}$ then we have 6 different equilateral triangles, 3 of type 5 and 3 more that are the mirror reflections of the triangles of type 5 in the indicated line. If $\ell = \sqrt{7}$ then we have 2 equilateral triangles, one of type 6 and the other the reflection of the triangle in the indicated line.



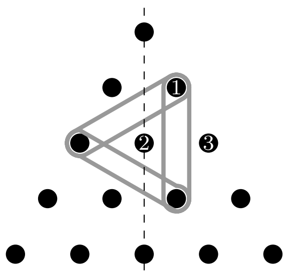
Type 2b



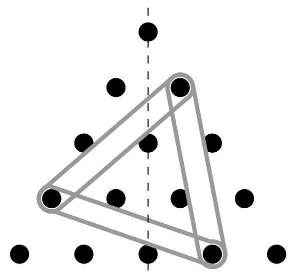
Type 3



Type 4



Type 5



Type 6

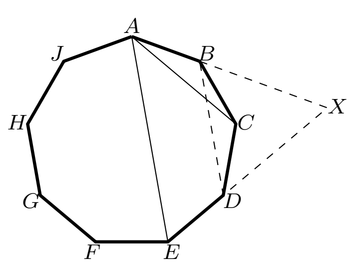
Therefore, in total we have 35 different equilateral triangles.

Also solved by IVAN GERGANOV, student, Kardzhali, Bulgaria; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; JUZ'AN NARI HAIFA, student, SMPN 8, Yogyakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MUHAMMAD LABIB IRFANUDDIN, student, SMP N 8 YOGYAKARTA, Indonesia; MUHAMMAD ROIHAN MUNAJIH, SMA Semesta Bilingual Boarding School, Indonesia; TAUPIEK DIDA PALLEVI, student, SMP N 8 YOGYAKARTA, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; KARTIKA CANDRA PUSPITA, student, SMPN 8, Yogyakarta, Indonesia; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; ARIF SETYAWAN, student, SMP N 8 YOGYAKARTA, Indonesia; and STEFANUS RENALDI WIJAYA, student, SMPN 8, Yogyakarta, Indonesia.

M514. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Nonagon $ABCDEFGHI$ is regular. Prove that $AE - AC = AB$.

I. Solution by Juz'an Nari Haifa, student, SMPN 8, Yogyakarta, Indonesia.



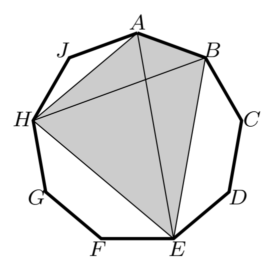
Extending AB and ED and let their point of intersection be X and draw BD. Since $ABCDEFGHI$ is regular, each of its interior angles is $140°$ and each of its exterior angles is $40°$. Clearly ΔBCD is isosceles, and hence $\angle CBD = \angle CDB = \dfrac{180° - 140°}{2} = 20°$. Hence $\angle XBD = \angle XDB = 60°$ and hence triangles XBD and XAE are equilateral. Since $ABCDEFGHI$ is regular, $AC = BD$ and thus

$$AE = AX = AB + BX = AB + BD = AB + AC$$

and therefore $AE - AC = AB$, as desired.

II. Solution by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania.

From Ptolemy's theorem applied to the cyclic quadrilateral $ABEH$ we get



$$AE \cdot BH = AH \cdot BE + AB \cdot EH\,.$$

Since $ABCDEFGHI$ is regular, $BH = BE = EH$ and $AH = AC$, so

$$AE \cdot BE = AC \cdot BE + AB \cdot BE\,.$$

thus

$$(AE - AC) \cdot BE = AB \cdot BE\,.$$

whence $AE - AC = AB$.

III. Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let O and R, respectively, be the circumcentre and circumradius of $ABCDEFGHI$. Since the chords AB, AC and AE, respectively, subtend angles of $40°$, $80°$ and $160°$ at O, then $AB = 2R\sin 20°$, $AC = 2R\sin 40°$, and $AE = 2R\sin 80°$. Hence

$$\begin{aligned} AE - AC &= 2R(\sin 80° - \sin 40°) \\ &= 2R\left(2\cos\frac{80° + 40°}{2}\sin\frac{80° - 40°}{2}\right) \\ &= 2R\sin 20° \\ &= AB \end{aligned}$$

as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (3 solutions); FLORENCIO CANO VARGAS, Inca, Spain; IOAN VIOREL CODREANU, Secondary School student, Satulung, Maramureş, Romania; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; RICARD PEIRÓ, IES "Abastos", Valencia, Spain (2 solutions); BRUNO SALGUEIRO FANEGO, Viveiro, Spain; MIHAÏ STOËNESCU, Bischwiller, France; DANIEL VĂCARU, Piteşti, Romania; and the proposer.

M515. *Proposed by Titu Zvonaru, Comăneşti, Romania.*

Without using calculus, determine the minimum and maximum values of

$$\frac{2x}{x^2+2x+2}$$

where x is a real number.

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Since $x^2+2x+2 = (x+1)^2+1 > 0$ for all values of x, then if x is nonnegative the fraction is nonnegative and if x is nonpositive the fraction is nonpositive. Thus the maximum occurs when $x \geq 0$ and the minimum when $x \leq 0$.

Let $x > 0$. Since $x + \frac{2}{x} \geq 2\sqrt{2}$ with equality if and only if $x = \frac{2}{x}$, then we have

$$\frac{2x}{x^2+2x+2} = \frac{2}{x+\frac{2}{x}+2} \leq \frac{2}{2+2\sqrt{2}} = \sqrt{2}-1$$

with equality if and only if $x = \sqrt{2}$, thus $\sqrt{2}-1$ is the maximum.

As for the minimum, say m, we write for $x \leq 0$,

$$\frac{2x}{x^2+2x+2} \geq m \iff \frac{2(-x)}{x^2+2x+2} \leq -m \iff \frac{2}{(-x)-2+\frac{2}{-x}} \leq -m$$

Denoting $-x = t \geq 0$ we get

$$\frac{2}{t-2+\frac{2}{t}} \leq \frac{2}{-2+2\sqrt{2}} = -m \Longrightarrow m = -1-\sqrt{2}$$

and

$$\left.\frac{2t}{t^2-2t+2}\right|_{t=\sqrt{2}} = 1+\sqrt{2}$$

The minimum is therefore $-1-\sqrt{2}$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ADRIENNA BINGHAM, Angelo State University, San Angelo, TX, USA; FLORENCIO CANO VARGAS, Inca, Spain; MARIUS DAMIAN, Brăila, Romania and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CORNELIU MĂNESCU-AVRAM, Transportation High School, Ploieşti, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; MIHAÏ STOËNESCU, Bischwiller, France; and the proposer.

M516. *Proposed by Syd Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Show that for any given nonzero integer k there exists at least four distinct ordered pairs (x, y) of integers such that

$$\frac{y^2-1}{x^2-1} = k^2-1.$$

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

If (x, y) is an ordered pair of integers such that $\dfrac{y^2-1}{x^2-1} = k^2 - 1$, then $(-x, y), (x, -y)$, and $(-x, -y)$ are ordered pairs which also satisfy the equation. These four ordered pairs are all distinct if x and y are nonzero. Hence it suffices to find an ordered pair of *positive* integers which satisfy the equation. Moreover, since $(-k)^2 = k^2$, we can suppose, without lose of generality, that k is a nonzero positive integer.

If $k = 1$, then $(x, y) = (2, 1)$ satisfies the equation. If $k = 2$ then $(x, y) = (3, 5)$ is a solution.

If $k \geq 3$ then $(x, y) = (k - 1, k^2 - k - 1)$ gives us:

$$\frac{y^2-1}{x^2-1} = \frac{(y+1)(y-1)}{(x+1)(x-1)} = \frac{k(k-1)(k+1)(k-2)}{k(k-2)} = k^2 - 1\,.$$

Since $k - 1$ and $k^2 - k - 1$ are both positive, then $(x, y) = (k - 1, k^2 - k - 1)$ is an ordered pairs of integers which satisfies the equation for each $k \geq 3$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ADRIENNA BINGHAM, Angelo State University, San Angelo, TX, USA; FLORENCIO CANO VARGAS, Inca, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CORNELIU MĂNESCU-AVRAM, Transportation High School, Ploieşti, Romania; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and the proposers.

M517. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all real solutions of the equation

$$3\sqrt{x+y} + 2\sqrt{8-x} + \sqrt{6-y} = 14.$$

Solution by Ricard Peiró, IES "Abastos", Valencia, Spain.

Let $8 - x = a^2$ and $6 - y = b^2$, where $a \geq 0$ and $b \geq 0$. Then we get $x + y = 14 - a^2 - b^2$. The initial equation becomes

$$3\sqrt{14 - a^2 - b^2} + 2a + b = 14$$
$$3\sqrt{14 - a^2 - b^2} = 14 - 2a - b\,.$$

Squaring both sides and rearranging yields

$$9(14 - a^2 - b^2) = 196 + 4a^2 + b^2 - 56a - 26b + 4ab$$

$$13a^2 + (4b - 56)a + 70 + 10b^2 - 28b = 0\,.$$

Solving the equation for a, we obtain

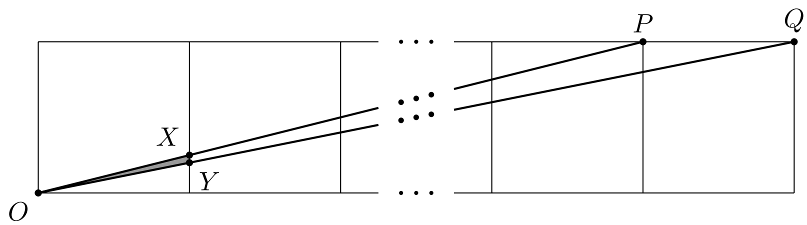
$$a = \frac{28 - 2b \pm \sqrt{126}\sqrt{-b^2 + 2b - 1}}{13}$$
$$= \frac{28 - 2b \pm \sqrt{126}\sqrt{-(b-1)^2}}{13}\,. \tag{1}$$

If $b-1 \neq 0$ then are no real solutions. Hence, the only real solution occurs when $b = 1$. From (1) we get $a = 2$. Since $8-x = 2^2$ and $6-y = 1^2$, the unique solution to the equation is $x = 4$ and $y = 5$.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; MARIUS DAMIAN, Brăila, Romania and NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania; JUZ'AN NARI HAIFA, student, SMPN 8, Yogyakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; CORNELIU MĂNESCU-AVRAM, Transportation High School, Ploieşti, Romania; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; DANIEL VĂCARU, Piteşti, Romania; and the proposer. One solution with no name was received.

M518. *Selected from a mathematics competition.*

A number of unit squares are placed in a line as shown in the diagram below.



Let O be the bottom left corner of the first square and let P and Q be the top right corners of the 2011^{th} and 2012^{th} squares respectively. When P and Q are connected to O they intersect the right side of the first square at X and Y respectively. Determine the area of triangle OXY.
(*This problem was inspired by question 2c from the 2004 Euclid Contest.*)

Solution by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania.

More generally, replace 2012 by an arbitrary natural number $n \geq 2$ and choose a system of rectangular coordinates with O at the origin, the x-axis containing the bases of the squares and the y-axis along a side of the first square. Consequently, we have $O(0,0), P(n-1,1)$ and $Q(n,1)$. Line OP has equation $(n-1)y = x$ and line OQ has equation $ny = x$, so that $X(1, \frac{1}{n-1})$ and $Y(1, \frac{1}{n})$. The area of the triangle OXY is equal to

$$\frac{y_X - y_Y}{2} = \frac{1}{2n(n-1)}\,.$$

For $n = 2012$, this area is equal to $\frac{1}{2\cdot 2011\cdot 2012}$.

Also solved by ADRIENNA BINGHAM, Angelo State University, San Angelo, TX, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; FLORENCIO CANO VARGAS, Inca, Spain; ANDHIKA GILANG, student, SMPN 8, Yogyakarta, Indonesia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MUHAMMAD ROIHAN MUNAJIH, SMA Semesta Bilingual Boarding School, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and DANIEL VĂCARU, Piteşti, Romania.

THE CONTEST CORNER

No. 12

Shawn Godin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_CCNuméro du problème *(exemple : Tremblay_Julie_CC1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions et réponses aux concours au rédacteur à l'adresse* `crux-contest@smc.math.ca`. *Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er mai 2014** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

CC56. On considère l'ensemble d'entiers consécutifs $\{1, 2, 3, \ldots, n\}$. On retire de cet ensemble trois entiers qui forment une suite géométrique. Les autres entiers de l'ensemble ont une somme de 6125. Déterminer la plus petite valeur possible de n, ainsi que toutes les suites géométriques correspondantes de trois termes qui ont pu être enlevées.

CC57. On considère un triangle acutangle DEF. On trace le cercle C_1 de diamètre DF et le cercle C_2 de diamètre DE. Les points Y et Z sont situés sur les côtés respectifs DF et DE de manière que les segments EY et FZ soient des hauteurs du triangle DEF. Le segment EY coupe le cercle C_1 en P, tandis que le segment FZ coupe le cercle C_2 en Q. Le prolongement de EY coupe le cercle C_1 en R, tandis que le prolongement de FZ coupe le cercle C_2 en S. Démontrer que les points P, Q, R et S sont cocycliques.

CC58. Déterminer toutes les valeurs réelles de x, y et z pour lesquelles

$$x - \sqrt{yz} = 42$$
$$y - \sqrt{zx} = 6$$
$$z - \sqrt{xy} = -30\,.$$

CC59. Neuf personnes s'exercent à exécuter une danse triangulaire, une danse qui regroupe trois personnes. À chaque exercice, les neuf personnes se regroupent en trois groupes de trois personnes et chaque groupe s'exerce indépendamment des deux autres. On considère que deux exercices sont différents s'il existe au moins une personne qui n'a pas dansé avec la même paire de personnes dans les deux exercices. Combien peut-il y avoir d'exercices différents ?

CC60. On considère l'équation

$$a_0^2 + a_0a_1 + a_1^2 + a_1a_2 + \cdots + a_{2009}a_{2010} + a_{2010}^2 = 1.$$

Combien de solutions entières cette équation admet-elle ?

. . . .

CC56. From the set of consecutive integers $\{1, 2, 3, \ldots, n\}$, three integers that form a geometric sequence are deleted. The sum of the integers remaining is 6125. Determine the smallest value of n and all three-term geometric sequences that make this possible.

CC57. Triangle DEF is acute. Circle C_1 is drawn with DF as its diameter and circle C_2 is drawn with DE as its diameter. Points Y and Z are on DF and DE respectively so that EY and FZ are altitudes of ΔDEF. EY intersects C_1 at P, and FZ intersects C_2 at Q. EY extended intersects C_1 at R, and FZ extended intersects C_2 at S. Prove that P, Q, R, and S are concyclic points.

CC58. Find all real values of x, y and z such that

$$x - \sqrt{yz} = 42$$
$$y - \sqrt{zx} = 6$$
$$z - \sqrt{xy} = -30\,.$$

CC59. Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place?

CC60. How many integer solutions are there to

$$a_0^2 + a_0a_1 + a_1^2 + a_1a_2 + \cdots + a_{2009}a_{2010} + a_{2010}^2 = 1?$$

CONTEST CORNER SOLUTIONS

CC6. Determine all pairs of positive integers a and b for which

$$3^{x+a} + 2^{x+a} + 2^x = 2^{x+b} + 3^x$$

is satisfied for some integer x.
(*Inspired by question #6 part b) from the 2012 Euclid contest.*)

Two incomplete solutions were received. Both solvers determined that if $a = 2$ *and* $b = 5$*, then* $x = 3$ *is a solution to the problem (this corresponds to the problem from the Euclid contest). One of the solvers made no attempt to find other solutions or to show there were none. The other solver made an attempt to show there were no other solutions, but the argument was flawed.*

CC7. Let $U = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disc in the plane $\mathbb{R}^2$. A chord of U is naturally defined to be a chord of the unit circle with its distinct endpoints removed. Prove or disprove: there is a bijection $f : \mathbb{R}^2 \to U$ such that every straight line in $\mathbb{R}^2$ is mapped by f onto a chord of U.
(*Originally question #3 from the 2012 Science Atlantic Math Competition(Barry Monson).*)

No solutions were received.

CC8. To see who pays for a pizza, A and B play the following simple game. They shuffle a deck of cards, and then in turns draw cards. The first person to draw an ace pays for the pizza. If A draws first, what is the probability that he buys? (Express your answer as a fraction in lowest terms.)
(*Originally question #6 from the 2012 Science Atlantic Math Competition.*)

Solved by Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Kamesha Strong, student, Auburn University at Montgomery, Montgomery, AL, USA. We give the solution of Strong.

Let $P(A_k)$ be the probability that the first ace drawn is by player A on his k^{th} draw. This occurs when the first $2(k-1)$ cards drawn are not aces and the next card is an ace. Since there are 4 aces in the deck, the first ace must be drawn by the 49th card, so for $k = 1, 2, \ldots, 25$

$$\begin{aligned} P(A_k) &= \frac{{}_{48}P_{2(k-1)}}{{}_{52}P_{2(k-1)}} \cdot \frac{4}{52 - 2(k-1)} \\ &= \frac{4({}_{48}P_{2(k-1)})}{{}_{52}P_{2k-1}} \\ &= \frac{4({}_{52}P_{2(k+1)})}{({}_{52}P_{2k-1})({}_{52}P_4)} \\ &= \frac{4({}_{(51-2k)}P_3)}{{}_{52}P_4}. \end{aligned}$$

And the probability that A buys the pizza is:

$$\begin{aligned} \sum_{k=1}^{25} P(A_k) &= \sum_{k=1}^{25} \frac{4({}_{(51-2k)}P_3)}{{}_{52}P_4} \\ &= \frac{4}{52 \cdot 51 \cdot 50 \cdot 49} \sum_{k=1}^{25} {}_{(2k+1)}P_3 \\ &= \frac{1}{26 \cdot 51 \cdot 25 \cdot 49} \sum_{k=1}^{25} (8k^3 - 2k) \\ &= \frac{1}{26 \cdot 51 \cdot 25 \cdot 49} \left[8\left(\frac{25 \cdot 26}{2}\right)^2 - 2\left(\frac{25 \cdot 26}{2}\right) \right] \\ &= \frac{1}{51 \cdot 49} [2 \cdot 25 \cdot 26 - 1] \\ &= \frac{433}{833}. \end{aligned}$$

[*Ed.: Alternately, we can consider the deck consisting of* 4 *cards that are aces, and* 48 *other cards. There* $\binom{52}{4}$ *ways to decide which* 4 *positions the aces are in, each of which is equally likely, so it suffices to determine for which of these configurations* A *will be the first player to draw an ace.*

We consider a bijection from these configurations to themselves formed by swapping the cards in positions $(2i-1, 2i)$ *for* $1 \leq i \leq 26$. *For example, the configuration* $C_1, C_2, C_3, C_4, \ldots, C_{51}, C_{52}$ *would become* $C_2, C_1, C_4, C_3, \ldots, C_{52}, C_{51}$. *Notice that after this bijection, a configuration for which* B *would draw the first ace becomes a configuration for which* A *would draw the first ace. A configuration for which* A *would draw the first ace becomes a configuration for which* B *would draw the first ace unless for some* i, *there was an ace in positions* $2i-1$ *and* $2i$ *and no ace in any positions before* $2i-1$.

We determine the probability that the first two aces occur in positions $2i-1$ *and* $2i$. *For a particular value of* i *there are* $\binom{52-2i}{2}$ *ways that this can occur, so* *the probability is*

$$\begin{aligned}\sum_{i=1}^{25}\frac{\binom{52-2i}{2}}{\binom{52}{4}} &= \sum_{i=1}^{25}\frac{\binom{2i}{2}}{\binom{52}{4}}\\ &= \frac{1}{\binom{52}{4}}\sum_{i=1}^{25}(2i^2-i)\\ &= \frac{1}{\binom{52}{4}}\left[2\left(\frac{25\cdot 26\cdot 51}{6}\right)-\frac{25\cdot 26}{2}\right]\\ &= \frac{10725}{270725}\\ &= \frac{33}{833}\,.\end{aligned}$$

This tells us that the probability A draws the first ace is

$$\frac{33}{833}+\frac{1}{2}\left(1-\frac{33}{833}\right)=\frac{433}{833}\ .]$$

CC9. Let $k \geq 3$ be an integer. Let $n = \frac{k(k+1)}{2}$. Let $S \subset \mathbb{Z}_n$ with $\|S\| = k$. Show that $S + S \neq \mathbb{Z}_n$. Note that $\|S\|$ denotes the cardinality of S and $S + S = \{x + y \mid x \in S, y \in S\}$.
(*Originally question #4 from the 2012 University of Waterloo Special K Contest.*)

Solution by Florencio Cano Vargas, Inca, Spain, modified by the editor.

There are n elements in $\mathbb{Z}_n$. If we choose a subset $S \subset \mathbb{Z}_n$ with k elements, $(a_1, a_2, \ldots, a_k)$, then the maximum number of elements in $S + S$ (if all sums are unique), is:

$$\binom{k}{2}+k=\binom{k+1}{2}=\frac{k(k+1)}{2}=n\,.$$

Since $S + S$ and $\mathbb{Z}_n$ could have the same cardinality, to prove that $S + S \neq \mathbb{Z}_n$ it is sufficient to prove that there are "repeated elements" in $S + S$, i.e., there exists two distinct subsets of S, $\{a_i, a_s\} \neq \{a_r, a_j\}$, such that $a_i + a_s = a_r + a_j$.

To prove this let us consider the set

$$S - S = \{\alpha_{ij} = a_i - a_j \mid a_i \in S, a_j \in S, a_i \neq a_j\} \subset \mathbb{Z}_n$$

Since we have imposed that $i \neq j$ then $\alpha_{ij} \neq 0$. There is a total of $k(k-1)$ distinct ordered pairs of elements from S so the cardinality of $S - S$ is at most $k(k-1)$. For $k \geq 3$ this is larger than $n - 1$. Indeed:

$$\begin{aligned}k(k-1) > n-1 &\Leftrightarrow k(k-1) > \frac{k(k+1)}{2}-1\\ &\Leftrightarrow 2k(k-1) > k^2+k-2\\ &\Leftrightarrow k^2-3k+2 > 0\end{aligned}$$

which is always true for $k \geq 3$.

This means that, for $k \geq 3$, by the pigeonhole principle, there exist at least two unique pairs of elements of S whose differences are equal. That is, there will be some elements α_{ij}, α_{rs} with $i \neq j, r \neq s$, and $i \neq r, j \neq s$ such that

$$\begin{aligned} \alpha_{ij} &= \alpha_{rs} \\ a_i - a_j &= a_r - a_s \\ a_i + a_s &= a_r + a_j \,. \end{aligned}$$

So we have proved that there are "repeated elements" in $S+S$, so $\|S+S\| < n = \|\mathbb{Z}_n\|$ and hence $S+S \neq \mathbb{Z}_n$.

CC10. Given a positive integer m, let $d(m)$ be the number of positive divisors of m. Determine all positive integers n such that $d(n) + d(n+1) = 5$.
(*Originally question #2 from the 2012 Sun Life Financial Repêchage Competition.*)

Solved by Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Mihaï-Ioan Stoënescu, Bischwiller, France; Daniel Văcaru, Piteşti, Romania; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania.

We present the solution given by Cano.

Let us consider separately the case $n = 1$. In this case $d(1) + d(2) = 3$ which is not a solution to the problem. Therefore we have to consider $n > 1$.

Let $d'(n)$ represent the number of non-trivial divisors of n, that is, the divisors excluding 1 and n. The condition of the problem can be rewritten as:

$$d'(n) + d'(n+1) = 1 \,.$$

If $n = 2k$ for some $k \geq 3$, then since $2 \mid n$ and $k \mid n$, $d'(n) \geq 2$. Since either n or $n+1$ is even, then $n+1 < 6$, and a quick check yields two possible solutions:

$$n = 3, n+1 = 4; d(3) = 2; d(4) = 3 \quad \text{and} \quad n = 4, n+1 = 5; d(4) = 3; d(5) = 2\,.$$

[*Ed.: Alternately, after disposing of the case $n = 1$, note that $d(n) \geq 2$ when $n > 1$, so the only possibilities are n such that $d(n) = 2$ and $d(n+1) = 3$, or $d(n) = 3$ and $d(n+1) = 2$. Note that $d(n) = 2$ if and only if n is prime, and $d(n) = 3$ if and only if $n = p^2$ for some prime p. Since n and $n+1$ are of opposite parity, the only possibilities are $n = 3$, a prime, with $n+1 = 4 = 2^2$; and $n = 4 = 2^2$, with $n+1 = 5$, a prime.*]

If you know of a mathematics contest at the high school or undergraduate level whose problems you would like to see in *Contest Corner*, please send information about the contest to `crux-contest@cms.math.ca`.

THE OLYMPIAD CORNER

No. 310

Nicolae Strungaru

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_OCNuméro du problème *(exemple : Tremblay_Julie_OC1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions et réponses aux concours au rédacteur à l'adresse* `crux-olympiad@smc.math.ca`*. Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er juin 2014** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de l'Université Saint-Boniface à Winnipeg, d'avoir traduit les problèmes.

OC116. Déterminer tous les entiers positifs n qui sont égaux à 300 fois la somme de leurs positions décimales.

OC117. Déterminer le plus petit entier positif m tel que pour tout nombre premier $p > 3$ on a

$$105 \mid 9^{p^2} - 29^p + m\,.$$

OC118. Déterminer toutes les fonctions $f : (0, \infty) \longrightarrow \mathbb{R}$ satisfaisant

$$f(x) + f(y) \le \frac{f(x+y)}{2} \quad \text{et} \quad \frac{f(x)}{x} + \frac{f(y)}{y} \ge \frac{f(x+y)}{x+y}\,,$$

pour tout $x, y \in (0, \infty)$.

OC119. Soit $ABCD$ un quadrilatère convexe et P le point dintersection des droites AC et BD. Supposons que $AC + AD = BC + BD$. Démontrez que les bissectrices des angles internes de $\angle ACB$, $\angle ADB$ et $\angle APB$ se coupent en un point.

OC120. Soit $S_r(n) = 1^r + 2^r + \cdots + n^r$ où n est un entier positif et r est un nombre rationnel. Le triplet (a, b, c) est dit élégant si a et b sont des nombres rationnels positifs, c est un entier positif et

$$S_a(n) = (S_b(n))^c$$

pour tout entier positif n. Déterminer tous les triplets élégants.

. . . .

OC116. Find all positive integers n which are 300 times the sum of their digits.

OC117. Find the smallest positive integer m such that for all prime numbers $p > 3$,

$$105 \mid 9^{p^2} - 29^p + m\,.$$

OC118. Find all functions $f : (0, \infty) \longrightarrow \mathbb{R}$ satisfying

$$f(x) + f(y) \le \frac{f(x+y)}{2} \quad \text{and} \quad \frac{f(x)}{x} + \frac{f(y)}{y} \ge \frac{f(x+y)}{x+y}\,,$$

for all $x, y \in (0, \infty)$.

OC119. Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB, \angle ADB$ and $\angle APB$ meet at a common point.

OC120. Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where n is a positive integer and r is a rational number. (a, b, c) is called a nice triple if a, b are positive rationals, c is a positive integer and

$$S_a(n) = (S_b(n))^c$$

for all positive integers n. Find all nice triples.

OLYMPIAD SOLUTIONS

OC56. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a function so that for all $a, b \in \mathbb{N}$ the expression $af(a) + bf(b) + 2ab$ is a perfect square. Prove that $f(a) = a$ for all $a \in \mathbb{N}$.

(*Originally question 3 from Iran Team Selection Test, Day 4, 2011.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

If p is an odd prime, then setting $a = b = p$ we get that $2p(p + f(p))$ is a perfect square and hence $p \mid p + f(p)$. Thus, for all odd primes p we have

$$p \mid f(p)\,.$$

Now let $p > f(1) - 2f(2)$ be an odd prime. Substituting $a = p$, $b = 1$ and then $a = p$, $b = 2$ into the given relation, we get positive integers c, d so that

$$c^2 = pf(p) + f(1) + 2p\,,$$

and

$$d^2 = pf(p) + 2f(2) + 4p\,.$$

Hence

$$c^2 < c^2 + 2p + 2f(2) - f(1) = d^2\,,$$

and thus $c + 1 \leq d$. Then

$$2c < d^2 - c^2 = 2p + 2f(2) - f(1) \leq 2(p + f(2))\,.$$

As a consequence, we have

$$pf(p) < c^2 < (p + f(2))^2\,.$$

For sufficiently large p, we have $(p + f(2))^2 < 2p^2$. Thus, there exists a K so that for $p > K$ we have $f(p) < 2p$, and since $p \mid f(p)$ we get

$$f(p) = p \text{ for each prime } p > K\,.$$

Let a be any natural number. We are going to prove $f(a) = a$ by contradiction. Consider the cases $f(a) < a$ and $f(a) > a$ in succession.

First, suppose that $f(a) < a$. For each prime $p > K$, setting $b = p$, the number $af(a) + p^2 + 2ap$ is a perfect square that is less than the perfect square $a^2 + p^2 + 2ap = (a + p)^2$. Thus,

$$af(a) + p^2 + 2ap \leq (a + p - 1)^2\,,$$

so that it must be true that

$$2p + 2a - 1 = (a + p)^2 - (a + p - 1)^2 \leq a(a - f(a))\,,$$

which is false for sufficiently large p. This is a contradiction.

Next, suppose $f(a) > a$. For each prime $p > K$, setting $b = p$, the number $af(a) + p^2 + 2ap$ is a perfect square that is greater than the perfect square $a^2 + p^2 + 2ap = (a + p)^2$. Thus,

$$(a + p + 1)^2 \leq af(a) + p^2 + 2ap\,,$$

so that it must be true that

$$2p + 2a + 1 = (a + p + 1)^2 - (a + p)^2 \leq a(f(a) - a)\,,$$

which is false for sufficiently large p. This is a contradiction, which completes the proof.

OC57. Let ABC be a triangle and A', B',C' be the midpoints of BC, CA, AB respectively. Let P and P' be points in a plane such that $PA = P'A'$, $PB = P'B'$, $PC = P'C'$. Prove that all PP' pass through a fixed point.
(*Originally question 2 from Iran Team Selection Test 2011, Day 4.*)

Solved by Michel Bataille, Rouen, France ; Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Cománeşti, Romania. We give Bataille's writeup.

We use complex coordinates with the origin at the circumcentre of ΔABC and, without loss of generality, we assume the circumcircle has radius 1. We will denote the complex coordinate of the point M by the corresponding small letter m. Then, we have

$$a\bar{a} = b\bar{b} = c\bar{c} = 1\,.$$

The hypothesis $PA = P'A'$ writes as

$$(p-a)(\bar{p}-\bar{a}) = \left(p' - \frac{b+c}{2}\right)\left(\bar{p'} - \overline{\frac{b+c}{2}}\right),$$

or

$$2p'(\bar{b}+\bar{c}) + 2\bar{p'}(b+c) - 4p\bar{a} - 4\bar{p}a - b\bar{c} - \bar{b}c = 4|p'|^2 - 4|p|^2 - 2\,.$$

Similarly

$$2p'(\bar{c}+\bar{a}) + 2\bar{p'}(c+a) - 4p\bar{b} - 4\bar{p}b - c\bar{a} - \bar{c}a = 4|p'|^2 - 4|p|^2 - 2\,.$$

Hence, the difference of the last two equations yields

$$2p'(\bar{a}-\bar{b}) + 2\bar{p'}(a-b) + 4p(\bar{a}-\bar{b}) + 4\bar{p}(a-b) - c(\bar{a}-\bar{b}) - \bar{c}(a-b) = 0\,.$$

In an analogous way we get

$$2p'(\bar{b}-\bar{c}) + 2\bar{p'}(b-c) + 4p(\bar{b}-\bar{c}) + 4\bar{p}(b-c) - a(\bar{b}-\bar{c}) - \bar{a}(b-c) = 0\,.$$

By eliminating $\bar{p'}$ from the last two equations, an easy computation yields $p' + 2p = 3w$ where

$$w = \frac{1}{6} \cdot \frac{\bar{a}(b^2-c^2) + \bar{b}(c^2-a^2) + \bar{c}(a^2-b^2)}{\bar{a}(b-c) + \bar{b}(c-a) + \bar{c}(a-b)}\,.$$

It follows that $\overrightarrow{PP'} = 3\overrightarrow{PW}$, which means that PP' passes through W. As the point W depends only on A, B and C, this proves the claim.

OC58. Find the smallest n for which there exists polynomials $f_1(x), f_2(x), \ldots, f_n(x) \in \mathbb{Q}[x]$ such that

$$f_1^2(x) + f_2^2(x) + \cdots + f_n^2(x) = x^2 + 7\,.$$

(*Originally question 3 from British IMO selection 2011, Day 1.*)

Solved by Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA. There were two incorrect solutions.

The solution is $n = 5$.

The example $f_1(x) = x,\ f_2(x) = 2,\ f_3(x) = f_4(x) = f_5(x) = 1$ shows that $n \leq 5$.

To prove that this is the smallest, we use the following result from [1, p. 815]:

> *If F is a field with* $\mathrm{char}(F) \neq 2$*, then $X^2 + d$ can be written as the sum of n squares in $F(X)$ if and only if either -1 or d is the sum of $n - 1$ squares in F.*

As -1 cannot be the sum of squares in $\mathbb{Q}$ and 7 cannot be the sum of three rational squares, it follows that $n = 4$ cannot work.

References

[1] Olga Taussky, *Sums of squares*, American Mathematical Monthly 77, 805-830, 1970.

OC59. Let n be an odd positive integer such that both $\phi(n)$ and $\phi(n+1)$ are powers of two. Prove $n+1$ is a power of two or $n = 5$.
(*Originally question 2 from Serbian Math Olympiad 2011.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Curtis.

If $p^2 \mid n$ then $p \mid \phi(n)$ which is not possible as n is odd. Similarly, if p is odd, p^2 cannot divide $n + 1$. Thus we have

$$n = \prod_{i=1}^{r} p_i \,,$$

and

$$n + 1 = 2^{\beta} \prod_{j=1}^{s} q_j \,,$$

where $p_1, \ldots, p_r$ are pairwise distinct odd primes and $q_1, \ldots, q_s$ are pairwise distinct odd primes.

As $\phi(n) = (p_1 - 1) \cdots (p_r - 1)$ is a power of two, each p_i is of the form

$$p_i = 2^{a_i} + 1 \,,$$

for distinct positive integers $a_1, \ldots, a_r$. Similarly, each q_j is of the form

$$q_j = 2^{b_j} + 1 \,,$$

for distinct positive integers $b_1, \ldots, b_s$.

Without loss of generality we have

$$a_1 < a_2 < \cdots < a_r$$

and

$$b_1 < b_2 < \cdots < b_s\,.$$

We also have

$$n = \prod_{i=1}^{r}(2^{a_i}+1)\,,$$

and

$$n+1 = 2^{\beta}\prod_{j=1}^{s}(2^{b_j}+1)\,,$$

We now use the well known fact that if 2^a+1 is prime then $a = 2^k$ for some k. Thus, there exists k_i, l_j so that $a_i = 2^{k_i}$ and $b_j = 2^{l_j}$. Thus we have

$$1+\prod_{i=1}^{r}(2^{2^{k_i}}+1) = 2^{\beta}\prod_{j=1}^{s}(2^{2^{l_j}}+1)\,.$$

It follows that when we expand $\prod_{i=1}^{r}(2^{2^{k_i}}+1)$ and $\prod_{j=1}^{s}(2^{2^{l_j}}+1)$ we get a sum of distinct powers of 2:

$$\prod_{i=1}^{r}(2^{2^{k_i}}+1) = 1+\sum_{k=1}^{M}2^{u_k}\,,$$

and

$$\prod_{j=1}^{s}(2^{2^{l_j}}+1) = 1+\sum_{k=1}^{N}2^{v_k}\,,$$

where

- $1 \le u_1 < u_2 < \cdots < u_M$;
- $1 \le v_1 < v_2 < \cdots < v_N$;
- $N = 2^s - 1$;
- $M = 2^r - 1$;
- $u_1 = a_1 = 2^{k_1}$ and if $r \ge 2$ then $u_2 = a_2 = 2^{k_2}$;
- $v_1 = b_1 = 2^{l_1}$ and if $s \ge 2$ then $v_2 = b_2 = 2^{l_2}$;

and we have

$$1+\left(1+\sum_{k=1}^{M}2^{u_k}\right) = 2^{\beta}\left(1+\sum_{k=1}^{N}2^{v_k}\right)\,.$$

Let's note that each of M and N is either 0 or odd.

Case 1: $u_1 \geq 2$. In this case we have

$$2 + \sum_{k=1}^{M} 2^{u_k} = 2^{\beta} \left(1 + \sum_{k=1}^{N} 2^{v_k}\right) .$$

As the left hand side is 2 (mod 4) it follows that $\beta = 1$ and

$$\sum_{k=1}^{M} 2^{u_k} = \sum_{k=1}^{N} 2^{v_k+1} .$$

This implies $M = N$ and $u_k = 1 + v_k$ for each $1 \leq k \leq M$.

- If $M = N = 0$, then $r = s = 0$ implying that $n = 1$ and $n+1 = 2$, thus $n+1$ is a power of 2.
- If $M = N = 1$ then $a_1 = u_1 = 1 + v_1 = 1 + b_1$, thus

  $$2^{k_1} = 1 + 2^{l_1} .$$

  This implies $l_1 = 0, k_1 = 1$ and hence $p_1 = 5$ and $q_1 = 3$. Thus in this case we get $n = p_1 = 5$.
- If $M = N \geq 1$ then exactly like in the case $N = M = 1$ we get $p_1 = 5$ and $q_1 = 3$. Moreover, as $a_2 = u_2 = v_2 + 1 = b_2$ we also get $k_2 = 1, l_2 = 1$ which yields $p_2 = 5$, a contradiction.

Case 2: $u_1 = 1$. In this case we have

$$4 + \sum_{k=2}^{M} 2^{u_k} = 2^{\beta} \left(1 + \sum_{k=1}^{N} 2^{v_k}\right) .$$

We know that $u_k \geq k$.

- If $u_k = k$ for all $2 \leq k \leq M$ then we get

  $$2^{M+1} = 4 + \sum_{k=2}^{M} 2^{u_k} = 2^{\beta} \left(1 + \sum_{k=1}^{N} 2^{v_k}\right) .$$

  This implies that $N = 0$ and $\beta = M + 1$, thus

  $$n + 1 = 2^{\beta} .$$
- If $u_k > k$ some $2 \leq k \leq M$ then if k_0 is the smallest such k we have

  $$4 + \sum_{k=2}^{M} 2^{u_k} = 2^{k_0} + \sum_{k=k_0+1}^{M} 2^{u_k} ,$$

and

$$k_0 < u_{k_0+1} < u_{k_0+2} < \cdots < u_M \,.$$

Thus we get

$$2^{k_0} + \sum_{k=k_0+1}^{M} 2^{u_k} = 2^{\beta} + \sum_{k=1}^{N} 2^{v_k+\beta} \,.$$

Using the uniqueness of representation of an integer as sum of distinct powers of 2, as

$$k_0 < u_{k_0+1} < u_{k_0+2} < \cdots < u_M$$

and

$$\beta < v_1 + \beta < v_2 + \beta < \cdots < v_N + \beta$$

we get

$$\begin{aligned} k_0 &= \beta \\ u_{k_0+1} &= v_1 + \beta \\ &\vdots \\ u_{M-1} &= v_{N-1} + \beta \\ u_M &= v_N + \beta \end{aligned}$$

From the definition of u_M and v_N we have $u_M = a_1 + \cdots + a_r$ and $v_N = b_1 + \cdots + b_s$. Moreover in this case we have $M \geq 2$ and $u_{M-1} = a_2 + a_3 + \cdots + a_r$ and either $N = s = 1$, in which case $u_{M-1} = \beta$ or $v_{N-1} = b_2 + \cdots + b_s + \beta$. In both situations, subtracting $u_M - u_{M-1}$ yields

$$a_1 = b_1 \,.$$

As $2^{a_1} + 1 \mid n$ and $2^{b_1} + 1 \mid n + 1$ it follows that $2^{a_1} + 1 \mid \gcd(n, n+1) = 1$ a contradiction.

This completes the proof.

OC60. On a blackboard we write the numbers $1, 2, \ldots, 20$. A move consists of selecting two numbers a, b from the blackboard so that $b \geq a + 2$, erasing a and b and writing instead $a+1$ and $b-1$. Find the maximum number of possible moves. (*Originally question 4 from Moldova Team Selection Test 2011, Day 2.*)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

The required number is 330.

We prove more generally that the maximum possible number of moves for the numbers $1, 2, \ldots, n$ on the blackboard is

$$\frac{1}{6} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n+1}{2} \right\rceil n. \tag{1}$$

Let $s = (s_1, \ldots, s_n)$ denote a state where s_k copies of the number k are on the blackboard, $s_k \geq 0$. The initial state is $S = (1, 1, \ldots, 1)$. We define the value of state s by $f(s) = \sum_{k=1}^{n} s_k k^2$, which is the sum of the squares of the numbers on the board. If a state s can be moved to a state t by selecting numbers a and b such that $a < b$, then

$$f(s) - f(t) = a^2 + b^2 - (a+1)^2 - (b-1)^2 = 2(b-a-1) \geq 2, \tag{2}$$

with equality if and only if $b = a + 2$.

By (2) and $f(t) > 0$, the process of making successive moves does eventually terminate at a final state T, say.

At the final state $T = (t_1, \ldots, t_n)$, the numbers on the blackboard are either identical or they take exactly two distinct neighboring values. Note that the average of the values on the blackboard is always $(n+1)/2$. Therefore, for an even number $n = 2m$, we have $t_m = t_{m+1} = m$ and $t_k = 0$ for $k \notin \{m, m+1\}$. For an odd number $n = 2m+1$, we obtain $t_{m+1} = n$ and $t_k = 0$ for $k \neq m+1$. By (2), the number $(f(S) - f(T))/2$ is an upper bound for the number of moves. For even $n = 2m$, we have

$$\frac{f(S) - f(T)}{2} = \frac{1}{2}\left(\sum_{k=1}^{2m} k^2 - m\left(m^2 + (m+1)^2\right)\right) = \frac{(m-1)(m+1)\cdot 2m}{6},$$

For odd $n = 2m+1$ we similarly obtain

$$\frac{f(S) - f(T)}{2} = \frac{m(m+1)(2m+1)}{6}.$$

This is the number (1) in either case.

We use the notation $s \underset{a}{\longrightarrow} t$ for a move with $b = a + 2$. For proving that the bound is attained, it is enough to show that there is a sequence of moves that transforms S to T such that only moves of the form $\underset{a}{\longrightarrow}$ are applied. Here it is:

$$\begin{aligned}
(\overbrace{1,1,\ldots,1}^{n}) &\underset{1}{\longrightarrow}\underset{2}{\longrightarrow}\cdots\underset{n-2}{\longrightarrow} (0,2,\overbrace{1,\ldots,1}^{n-4},2,0) \\
&\left(\underset{2}{\longrightarrow}\underset{3}{\longrightarrow}\cdots\underset{n-3}{\longrightarrow}\right)^2 (0,0,3,\overbrace{1,\ldots,1}^{n-6},3,0,0) \\
&\left(\underset{3}{\longrightarrow}\underset{4}{\longrightarrow}\cdots\underset{n-4}{\longrightarrow}\right)^3 (0,0,0,4,\overbrace{1,\ldots,1}^{n-8},4,0,0,0) \\
&\vdots
\end{aligned}$$

For even numbers $n = 2m$ it ends at

$$(\overbrace{0,\ldots,0}^{m-2},m-1,1,1,m-1,\overbrace{0,\ldots,0}^{m-2}) \left(\underset{m-1}{\longrightarrow}\right)^{m-1} (\overbrace{0,\ldots,0}^{m-1},m,m,\overbrace{0,\ldots,0}^{m-1}).$$

For odd numbers $n = 2m+1$ it ends at

$$(\overbrace{0,\ldots,0}^{m-1},m,1,m,\overbrace{0,\ldots,0}^{m-1}) \left(\underset{m}{\longrightarrow}\right)^{m} (\overbrace{0,\ldots,0}^{m},n,\overbrace{0,\ldots,0}^{m}).$$

This completes the proof that the bound (1) is attained.

BOOK REVIEWS

Amar Sodhi

Jim Totten's Problem of the Week edited by John Grant McLoughlin, Joseph Khoury and Bruce Shawyer
World Scientific Publishing, 2013
ISBN: 978-981-4513-30-2 Hardcover, 337+ix pages, US$68
Reviewed by ***Amar Sodhi****, Grenfell Campus, Memorial University of Newfoundland.*

Each week, during the Fall and Winter semesters, Jim Totten would post, on the bulletin board of the mathematics lab at Thompson Rivers University in Kamloops, B.C., a challenge. Namely, a mathematical poser chosen for the enjoyment of the undergraduate student body. This tradition lasted over twenty-five years, thereby leaving several hundred worthy problems for Grant McLoughlin, Khoury and Shawyer to select, categorize and publish in a single volume.

The name Totten should be familiar to many readers of *Crux Mathematicorum*. Jim Totten joined the editorial board of this journal in 1994. Bruce Shawyer was Editor-in-Chief of *Crux* from 1996 to 2002, a role which Jim undertook from 2003 until his untimely death in March 2008, just three months before his intended retirement from this post. John Grant McLoughlin was also on the editorial board of Crux during the Totten years and shared Jim's passion in mathematical outreach. It was fitting, therefore, that John would write an article about "Jim Totten's Reach" in the Totten Commemorative issue of *Crux* (issue 5 of volume 35 published September 2009).

Jim Totten's Problem of the Week is a collection which demonstrates Jim's desire to engage undergraduate students with an interest in mathematics. That means, each problem has the potential to pique the interest of both a freshman and senior undergraduate who will congratulate themselves upon reaching a correct solution; the level of difficulty of the problems suggest that those with an aptitude for mathematics will be successful more often than not. Most problems are posed for the mathematically literate audience and the chapter titles suggest as much. However, there are many brain teasers in this volume; enough to warrant a slim volume for a general audience.

Four hundred and six problems were selected for this book and distributed through ten chapters: Combinatorial Geometry; Functions; Higher Dimensional Geometry; Identities, Inequalities and Expressions; Logic, Games, Puzzles and Amusement in Math; Number Theory; Plane Geometry; Probability; Triangle Mathematics; Miscellaneous. Almost half these problems are contained in the chapters Logic, Games, Puzzles and Amusement in Math and Number Theory. The latter contains many problems of a combinatorial nature as well as problems which, a century ago, may well as been classified as arithmetic puzzles. As an example, problem 211 in the book is essentially asking the reader to find all arrangements of $\{1, 2, 3, 4, 5, 6\}$ which gives rise to a number where each of the first k digits is divisible by k. Totten defined such a number as cute and it is certainly a cute problem.

I would suspect that students would find the seventy odd problems in Plane Geometry and Triangle Mathematics the most challenging; one of the more arithmetic challenges in these chapters is problem 342 which asks the reader to prove that there exists no triangle with altitudes of 4, 7 and 10. A solver who is a whizz at trigonometry may enjoy problem 291 which requires one to find the length of BD in a quadrilateral $ABCD$ with $AB \parallel DC$, $BC = \sqrt{2}$ and $AB = AC = AD = \sqrt{3}$.

The book is also a valuable resource for instructors who like to give an occasional challenge to their students. Finding necessary and sufficient conditions on the coefficients on a quadratic which has a number and its square as roots (problem 60) might be just the problem to stimulate a keen student in a precalculus course.

There are two quibbles I have with the publisher of the book. On the back cover they claim that: "It is a resource for those interested in mathematical competitions ranging from high school level to the William Lowell Putman Mathematical Competition." While it is true that students who plan to write the Putman would find Jim Totten's Problem of the Week interesting, there are better resources, such as *Crux*, than this book. No, this is a book containing problems that students with little interest in competitions may well find enjoyable.

A more serious concern are the solutions to the problems. Each problem has its precise and well written solution appearing immediately after the problem; so only the most careful reader will avoid a glimpse of the solution prior to tackling the problem. Hopefully, in a second edition, the solutions to problems will be placed at the end of each chapter.

All things considered though, I heartily recommend this book to any instructor of mathematics, for it contains a wealth of problems to share with students. I would also recommend this book to anyone who enjoys a mathematical challenge which does not require any machinery from a senior level mathematics course, or even first year calculus.

PROBLEM SOLVER'S TOOLKIT

No. 4

J. Chris Fisher

The Problem Solver's Toolkit is a new feature in ***Crux Mathematicorum****. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

Harmonic Sets Part 1: Desargues Theorem

The goal of this multi-part essay is to study harmonic sets, but to reach that goal we shall first have to review several basic topics from projective geometry. Since our main interest here is Euclidean geometry, it will be convenient for us to obtain the projective plane from the Euclidean plane by adjoining a *line at infinity*, a line that by definition consists of exactly one new point on every line of the Euclidean plane; two lines of the Euclidean plane share one of these new points if and only if they are parallel. The points and lines of the extended plane form a model of the real projective plane; every pair of lines of the projective extension intersects in exactly one point. In other words we are assuming that the points and lines of our extended plane satisfy two simple properties:

Two points determine a line, on which they both lie;
Two lines determine a point, through which they both pass.

In this way we can speak of parallel lines as intersecting at infinity, thus avoiding annoying special cases in proofs of Euclidean theorems. As a bonus we inherit a duality principle: From any true statement involving points, lines, and incidence we obtain another true statement by mechanically interchanging "point" and "line", and making consequent linguistic changes such as switching "lie" with "pass" and "on" with "through."

We are concerned here with properties of configurations that consist of points and lines. Such properties will be preserved by projecting a configuration from one plane to another. The property that makes it all work was first observed by the seventeenth century French architect Girard Desargues. We say that two triangles ABC and $A'B'C'$ are *perspective from a centre* O if the lines joining corresponding points (namely AA', BB', and CC') are concurrent at O; dually, we say that the triangles are *perspective from an axis* o if the intersection points of corresponding sides (namely AB with $A'B'$ at C'', BC with $B'C'$ at A'', CA with $C'A'$ at B'') are collinear on o.

Theorem (Desargues). Two triangles are perspective from a point if and only if they are perspective from a line.

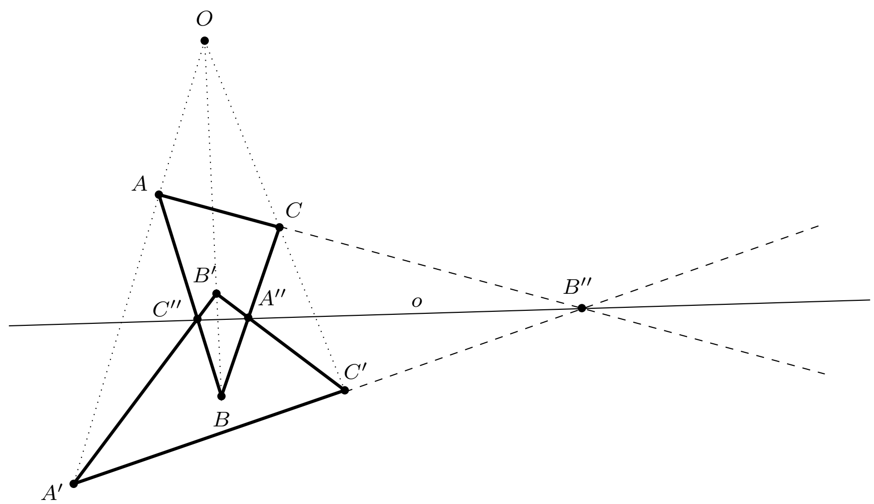


Figure 1: The triangles ABC and $A'B'C'$ are perspective from the point O and from the line o.

There are dozens of proofs of the theorem, some with algebra, some without, that can easily be found elsewhere. Here we will simply observe that the theorem is self dual: the "if" statement is the dual of the "only if" statement. Although we distinguished the centre and axis of the perspectivity in the statement of the theorem, when the ten points and lines of the theorem are considered to be the points and lines of a configuration, then any of the ten points could be chosen to be the centre. The accompanying figure shows the configuration with all ten of its points in the Euclidean plane. The configuration can have as many as four of its points at infinity. Starting with congruent triangles ABC and $A'B'C'$ that have their corresponding sides parallel, we get a Desargues configuration with four points at infinity; a pair of similar triangles with corresponding sides parallel leads to a configuration with three points at infinity.

Exercise 1. Draw a Desargues configuration with exactly one point at infinity. Draw another with exactly two points at infinity.

Exercise 2. Label the ten points of a Desargues configuration by unordered pairs (ij) of the integers from 1 to 5 using the rule that *the three pairs that can be formed from three integers must lie on the same line*. Dually, each line would be labeled by the two integers $[st]$ that do not appear in a label of any of its three points. So, for example, if we attach the pair (12) to A, the three lines through A will be assigned the pairs $[34], [35], [45]$ (in any order), while the remaining points on [34] get the labels (15) and (25). The rest of the labels are now easily determined by observing the rule that the symbols for the three vertices of each of the five triangles formed by points and lines of the configuration must have exactly one integer in common.

The exercise tells us that the automorphism group of a Desargues configuration is the permutation group S_5 of order 5!; in other words, there are 120 ways to label the vertices using the rules of exercise 2.

Exercise 3. How can a person plant ten trees in ten rows of three each?

Exercise 4. Prove that the medians of a triangle are concurrent.

Exercise 5. Two lines are drawn on a sheet of paper, but intersect at a point that is far off the sheet. Given a point on that page, draw the line that joins it to the inaccessible intersection of the given lines.

Exercise 6. Prove that if three triangles are perspective in pairs from the same axis, then their centres of perspective are collinear.

Projective geometry is concerned with collineations of the projective plane; these are the incidence preserving mappings that take points to points and lines to lines. The building blocks of the theory are the *perspective collineations*; these collineations fix all lines through a point (called the *centre*), and all the points on a line (called the *axis*). Familiar examples from Euclidean geometry include the *translations* with both centre and axis at infinity (represented using Cartesian coordinates by $(x, y) \to (x + a, y + b)$), the *dilatations* ($(x, y) \to (ax, ay)$) with centre at the origin and axis at infinity, the *strains* ($(x, y) \to (x, ay)$)) with axis $y = 0$ and centre at infinity, and *shears* $(x, y) \to (x + y, y)$ with axis $y = 0$ and centre at the point of infinity of the axis. In the projective plane, any line o can serve as axis and any point O as centre; the central collineation is then determined by any point A not on o and its image A', which can be any point except O on OA that is not on o. To find the image of any other point B in the plane, we know that if B is on o then B is fixed; otherwise,

(i) the image B' must lie on OB (by the definition of a central collineation), and
(ii) AB must intersect o in a fixed point, call it C'' (because in a projective plane, any two lines must intersect, and every point of o is fixed).

Because collineations preserve incidence, C'' (which is fixed) must lie on both AB and its image line $A'B'$, so that we must have $B' = OB \cap A'C''$. Similarly, for any point C not on OA or OB, if AC meets o at B'' then $C' = OC \cap A'B''$. But how do we know that the image C' of C is well defined by this process? How do we know that if we used B instead of A to define C' we would get the same point? Just look back at Figure 1 and you will see that the desired result (namely, that $BC \cap B'C'$ lies on o) is guaranteed by Desargues's theorem.

Exercise 7. Prove that the product of two perspective collineations that have the same axis is a perspective collineation unless it is the identity.

Six Ways to Count the Number of Integer Compositions

Amitabha Tripathi

Counting problems are at the core of the field of study that we call "Combinatorics". Basic principles and techniques of Combinatorics familiar to any undergraduate student, and often to an advanced high school student, include basic laws of sum and product, counting distributions, permutations and combinations, the principle of Induction, and the principle of Inclusion & Exclusion. A useful method often employed to count the number of objects in a set is to place the set in one-to-one correspondence with another set whose size is more easily determined. The problem of counting the number of compositions of a positive integer is standard, and can be found in some form in several books that deal with basic combinatorial methods. We illuminate this problem by touching upon many different basic counting methods that may be employed to solve this problem.

Let n and k be positive integers. A `composition` of n into k (positive) parts is an ordered k-tuple $(x_1, \ldots, x_k)$ with each $x_i \in \mathbb{N}$ and $x_1 + \cdots + x_k = n$. The x_i's are the `parts` of the composition. Thus $(1, 2, 2, 3)$ is a composition of 8 into 4 parts. If we denote by $p_k^\star(n)$ the number of compositions of n into k parts, it is a standard result in combinatorics that

$$p_k^\star(n) = \left|\left\{(x_1, \ldots, x_k) : x_1 + \cdots + x_k = n, x_i \geq 1\right\}\right| = \binom{n-1}{k-1}. \qquad (1)$$

It is worth recalling an ingenious method to solve this (and other similar) problems. Place n dashes on a line, with adjacent dashes separated by blank spaces. Note that there are $n-1$ blank spaces. Choose $k-1$ of these $n-1$ blank spaces and fill them with $k-1$ bars; this can be done in $\binom{n-1}{k-1}$ ways. Each of these ways breaks the n dashes into k nonempty batches, thus providing a (unique) composition of n into k parts. To see this, the figure below illustrates the example of $(1, 2, 2, 3)$ as a composition of 8 into 4 parts by placing bars into spaces 1, 3, and 5.

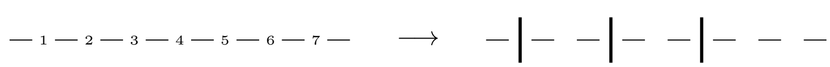


Figure 2: $(1, 2, 2, 3)$ as a composition of 8 into 4 parts

The selection of $k-1$ of the $n-1$ blank spaces to be filled with bars is associated in a one-to-one manner with the composition of n into k parts. Thus there are $\binom{n-1}{k-1}$ compositions of n into k parts.

A `composition` of n is a composition of n into k parts, with no restriction on the number of parts. We list below the number of compositions of n.

n	1	2	3	4	5	6	7	8	9	10
$p^\star(n)$	1	2	4	8	16	32	64	128	256	512

The pattern is unmistakeable, and suggests a simple combinatorial explanation. Indeed, if $p^\star(n)$ denotes the number of compositions of n, then

$$p^\star(n) = \sum_{k\geq 1} p^\star_k(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}. \tag{2}$$

The method of placing bars between dashes to obtain a formula for $p^\star_k(n)$ can be easily adapted to also obtain the result in (2). In fact, since we have no restriction on the number of bars to place between the dashes, there are exactly 2 choices (whether or not to place a bar) between each pair of adjacent dashes. Each of these choices can again be associated in a one-to-one manner with the compositions of n, leading to 2^{n-1} compositions of n.

The simplicity of formula (2) for $p^\star(n)$ leads us to believe that there may be other ways to obtain this result by using other basic combinatorial methods. Underlying this is the understanding that any two finite sets of the same size must be in one-to-one correspondence, and it is this correspondence that we seek.

Induction. One of the first thoughts that cross our minds when we need to prove a formula that applies to the set of positive integers is to apply the method of mathematical induction. Note that $p^\star(1) = 2^0 = 1$. Assume that $p^\star(m) = 2^{m-1}$ for $1 \leq m \leq n-1$. Each composition of n begins with a k for some k with $1 \leq k \leq n$. Since there are $p^\star(n-k)$ compositions of n with first part k, $1 \leq k < n$, and one with first part n, using induction hypothesis we get

$$p^\star(n) = 1 + \sum_{k=1}^{n-1} p^\star(n-k) = (2^{n-2} + 2^{n-3} + \cdots + 2 + 1) + 1 = 2^{n-1}.$$

Recurrence. One of the simplest, yet powerful, methods to resolve a combinatorial problem is to find a recurrence equation satisfied by the function that solves the problem, then use standard methods to solve the recurrence. Recurrences are not easy to resolve in general, yet the sequence $\{p^\star(n)\}_{n\geq 1}$ looks promising given the sequence it represents.

The set of compositions of n with first part 1 is in one-to-one correspondence with the set of compositions of $n-1$ via $(1, a_2, a_3, \ldots, a_k) \leftrightarrow (a_2, a_3, \ldots, a_k)$. The set of compositions of n with first part $m > 1$ is in one-to-one correspondence with the set of compositions of $n-1$ with first part $m-1$ via $(a_1, a_2, \ldots, a_k) \leftrightarrow (a_1 - 1, a_2, \ldots, a_k)$. The latter set is the set of all compositions of $n-1$. Hence $p^\star(n) = 2 \cdot p^\star(n-1)$ for $n \geq 2$, and since $p^\star(1) = 2^{1-1}$, we have $p^\star(n) = 2^{n-1}$.

Generating Functions. Several counting problems are easily resolved by evaluating or simplifying their generating function. This is particularly true of functions that are linear combinations of geometric sequences.

Since $p^\star_k(n)$ counts the number of k-tuples $(a_1, \ldots, a_k)$ of positive integers whose sum is n, it equals the coefficient of x^n in the expansion $(x+x^2+x^3+\cdots)^k =$

$x^k(1-x)^{-k}$. Hence the sequence $\{p^\star(n)\}_{n\geq 1}$ has the generating function

$$\sum_{n=1}^{\infty} p^\star(n)x^n = \sum_{k=1}^{\infty} x^k(1-x)^{-k} = \frac{x}{1-2x} = x\sum_{n=0}^{\infty}(2x)^n.$$

Comparing coefficients of x^n, we have $p^\star(n) = 2^{n-1}$.

Sets. We know that if S has n elements, then the power set $\mathcal{P}(S)$ of S has 2^n elements. Therefore it must be the case that the power set of a set of size $n-1$ must be in one-to-one correspondence with the set of compositions of n, given the formula in (2). We exploit this fact to produce such a one-to-one correspondence between the two sets.

Write $\mathcal{P}^\star(n)$ for the set of all ordered tuples $(a_1,\ldots,a_k)$ of positive integers whose sum is n, $\mathcal{P}(S)$ for the power set of S, and $[m]$ for $\{1,\ldots,m\}$ where $m\in\mathbb{N}$. Define a mapping $\varphi : \mathcal{P}^\star(n) \to \mathcal{P}\big([n-1]\big)$ by

$$(a_1,\ldots,a_k) \mapsto \{a_1, a_1+a_2, \ldots, a_1+a_2+\cdots+a_{k-1}\},$$

with $(n)\mapsto\emptyset$.

We show that φ is *one-one* and *onto*. Suppose $\mathbf{a} := (a_1,\ldots,a_r)$ and $\mathbf{b} := (b_1,\ldots,b_s)$ are unequal elements in $\mathcal{P}^\star(n)$. So $a_i = b_i$ for $1\leq i<j$, and $a_j<b_j$ for a suitable j. Then $a_1+\cdots+a_j \in \varphi(\mathbf{a})\setminus\varphi(\mathbf{b})$ since

$$b_1+\cdots+b_{j-1} = a_1+\cdots+a_{j-1} < a_1+\cdots+a_{j-1}+a_j < b_1+\cdots+b_{j-1}+b_j,$$

and the elements of $\varphi(\mathbf{b})$ are arranged in increasing order. Hence φ is one-one.

Any non-empty subset $\{b_1,\ldots,b_k\}$ of $[n-1]$, with $b_1<\cdots<b_k$, is the image of $(b_1, b_2-b_1, b_3-b_2, \ldots, b_k-b_{k-1}, n-b_k)\in\mathcal{P}^\star(n)$, as can be easily verified. Hence φ is also onto.

Functions. It is customary for B^A to denote the set of all functions with domain A and codomain B, since $\left|B^A\right| = |B|^{|A|}$ for finite sets A, B. So if we denote by $\mathbf{2}$ the set $\{0,1\}$, by $\mathbf{2}^S$ we mean the set of all functions from S into $\mathbf{2}$, and this set has size $|\mathbf{2}|^{|S|} = 2^{|S|}$. Borrowing the notations from Sets, we must now exhibit a one-to-one correspondence between the sets $\mathcal{P}^\star(n)$ and $\mathbf{2}^{[n-1]}$. Recall the characteristic function of a set A is a function χ_A defined as

$$\chi_A(a) = \begin{cases} 1 & \text{if } a\in A \\ 0 & \text{if } a\notin A \end{cases}.$$

Define a mapping $\psi : \mathcal{P}^\star(n) \to \mathbf{2}^{[n-1]}$ by

$$(n)\mapsto \chi_\emptyset = \mathbf{0}, \quad (a_1,\ldots,a_k)\mapsto\chi_A,$$

where $A = \{a_1, a_1+a_2, \ldots, a_1+a_2+\cdots+a_{k-1}\}$.

We show that ψ is *one-one* and *onto*. With the notations used for Sets, recall that $a_1 + \cdots + a_j \in \varphi(\mathbf{a}) \setminus \varphi(\mathbf{b})$. So if $A = \{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{r-1}\}$ and $B = \{b_1, b_1 + b_2, \ldots, b_1 + b_2 + \cdots + b_{s-1}\}$, then $\chi_A \neq \chi_B$. Hence ψ is one-one.

Suppose $f \in \mathbf{2}^{[n-1]}$, with $f \neq \mathbf{0}$. Let $f^{-1}(1) = \{a_1, \ldots, a_k\}$, where $a_1 < \ldots < a_k$. Then it is easy to verify that f is the image under ψ of $(a_1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}, n - a_k) \in \mathcal{P}^\star(n)$. Hence ψ is also onto.

Inclusion & Exclusion. The Principle of Inclusion & Exclusion (PIE) is useful in solving such diverse problems as counting the number of derangements, the number of onto mappings between two finite sets, and number of integers relatively prime to and less than a given integer. PIE counts the number of elements outside of finitely many (finite) sets, giving the result in terms of intersections of these sets. When only two sets are involved, PIE is just the formula

$$|A \cup B| + |A \cap B| = |A| + |B|,$$

for *finite* sets A, B. We include this connection as our last example, but this connection also relies on induction.

We also induct on n. Observe that $p^\star(1) = 1$, and assume that $p^\star(m) = 2^{m-1}$ for $1 \leq m \leq n$. To each $\mathbf{a} = (a_1, \ldots, a_k) \in \mathcal{P}^\star(n)$, we associate two elements of $\mathcal{P}^\star(n+1)$, given by

$$\ell(\mathbf{a}) := (1, a_1, \ldots, a_k), \qquad r(\mathbf{a}) := (a_1, \ldots, a_k, 1).$$

Define

$$\begin{aligned} \mathcal{L}^\star(n) &:= \ell\Big(\mathcal{P}^\star(n)\Big) = \{\ell(\mathbf{a}) : \mathbf{a} \in \mathcal{P}^\star(n)\}, \\ \mathcal{R}^\star(n) &:= r\Big(\mathcal{P}^\star(n)\Big) = \{r(\mathbf{a}) : \mathbf{a} \in \mathcal{P}^\star(n)\}; \\ \mathcal{Q}^\star(n) &:= \mathcal{P}^\star(n) \setminus \Big\{\mathcal{L}^\star(n-1) \cup \mathcal{R}^\star(n-1)\Big\}. \end{aligned}$$

Therefore

$$|\mathcal{P}^\star(n)| = |\mathcal{L}^\star(n-1) \cup \mathcal{R}^\star(n-1)| + |\mathcal{Q}^\star(n)| \tag{3}$$

Clearly $|\mathcal{L}^\star(n)| = |\mathcal{P}^\star(n)| = |\mathcal{R}^\star(n)|$, and these equal 2^{n-1} by induction hypothesis. We claim that $|\mathcal{L}^\star(n-1) \cap \mathcal{R}^\star(n-1)| = 2^{n-2}$. Any $\mathbf{a} \in \mathcal{L}^\star(n-1) \cap \mathcal{R}^\star(n-1)$ must have first and last part 1. Removing these 1's results in an element in $\mathcal{P}^\star(n-1)$. Conversely, to any element in $\mathcal{P}^\star(n-1)$ we can attach a 1 at both ends and obtain an element in $\mathcal{L}^\star(n-1) \cap \mathcal{R}^\star(n-1)$. Hence $\mathcal{L}^\star(n-1) \cap \mathcal{R}^\star(n-1)$ is in one-to-one correspondence with $\mathcal{P}^\star(n-1)$, and so by induction hypothesis $|\mathcal{L}^\star(n) \cap \mathcal{R}^\star(n)| = 2^{n-2}$. Therefore

$$\begin{aligned} |\mathcal{L}^\star(n) \cup \mathcal{R}^\star(n)| &= |\mathcal{L}^\star(n)| + |\mathcal{R}^\star(n)| - |\mathcal{L}^\star(n) \cap \mathcal{R}^\star(n)| \\ &= 2^{n-1} + 2^{n-1} - 2^{n-2} = 3 \cdot 2^{n-2} \end{aligned} \tag{4}$$

Note that $\mathbf{a} = (a_1, \ldots, a_k) \in \mathcal{Q}^\star(n) \subset \mathcal{P}^\star(n)$ if and only if $a_1 > 1$ and $a_k > 1$. The correspondence $(a_1, \ldots, a_k) \leftrightarrow (a_1 - 1, a_2, \ldots, a_{k-1}, a_k - 1)$ sets up a one-to-one correspondence between $\mathcal{Q}^\star(n)$ and $\mathcal{P}^\star(n-2)$. Therefore

$$|\mathcal{Q}^\star(n)| = 2^{n-3} \tag{5}$$

From equations (3), (4), (5) we get

$$p^\star(n+1) = |\mathcal{P}^\star(n+1)| = |\mathcal{L}^\star(n) \cup \mathcal{R}^\star(n)| + |\mathcal{Q}^\star(n+1)| = (3 \cdot 2^{n-2}) + 2^{n-2} = 2^n.$$

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PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. Nous préférons les réponses électroniques et demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats (Word, etc.) soient aussi acceptés. Nous invitons les lecteurs à envoyer leurs solutions au rédacteur à l'adresse* crux-redacteurs@smc.math.ca*. Nous acceptons aussi les contributions par la poste, envoyées à l'adresse figurant en troisième de couverture. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'tablissement qu'elle fréquente, sa ville et son pays; chaque solution doit également commencer sur une nouvelle page. Un astérisque (⋆) signale un problème proposé sans solution.*

Nous sommes surtout à la recherche de problèmes originaux, mais d'autres problèmes intéressants peuvent aussi être acceptables pourvu qu'ils ne soient pas trop connus et que leur provenance soit indiquée. Normalement, si l'on connaît l'auteur d'un problème, on ne doit pas le proposer sans lui en demander la permission. Les solutions connues doivent accompagner les problèmes proposés. Si la solution n'est pas connue, la personne qui propose le problème doit tenter de justifier l'existence d'une solution. Il est recommandé de nommer les fichiers de la manière suivante : Nom de famille_Prénom_Proposition_Année_numéro *(exemple : Tremblay_Julie_Proposition_2014_4.tex, s'il s'agit du 4e problème proposé par Julie en 2014).*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er juin 2014***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal d'avoir traduit les problèmes.

3793. *Correction. Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b et c trois nombres réels positifs tels que

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=1007\sqrt{2}\ .$$

Trouver la valeur minimale de l'expression

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\ .$$

3811. *Proposé par Jung In Lee, Seoul Science High School, Seoul, Republic of Korea.*

Déterminer toutes les fonctions $f:\mathbb{N}\to\mathbb{N}$ telles que, pour tous les entiers positifs a et b, $af(a+b)+bf(a)+b^2$ soit un carré parfait.

3812. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit $ABCD$ un parallélogramme et P un point sur le côté BC. Soit respectivement K, L et M les centres de gravité des triangles PAB, PAD et PCD. Montrer que

$$[AKL] + [DLM] = [BKMC],$$

où $[\cdot]$ représente l'aire.

3813. *Proposé par Michel Bataille, Rouen, France.*

Trouver la plus petite constante C telle que l'inégalité

$$(a^7 + b^7 + c^7)^6 \leq C(a^6 + b^6 + c^6)^7$$

soit valide pour tous les nombres réels a, b, c tels que $a + b + c = 0$.

3814. *Proposé par Marcel Chiriţă, Bucarest, Roumanie.*

Montrer que pour tout nombre x dans l'intervalle fermé $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$, il existe un point M dans le plan du carré $ABCD$ tel que

$$x = \frac{AM + MC}{BM + MD}.$$

3815. *Proposé par Paolo Perfetti, Département de Mathématiques, Université de Rome, "Tor Vergata", Rome, Italie.*

Montrer que $x^x \leq x^2 - x + 1$ pour tous les x avec $0 \leq x \leq 1$.

3816. *Proposé par Mehmet Şahin, Ankara, Turquie.*

Soit ABC un triangle rectangle avec l'angle droit en C, et soit D le pied de la hauteur issue de C. Soit respectivement I_1 et I_2 les centres des cercles inscrits des triangles CAD et CBD. Soit ρ et r les rayons de ceux des triangles I_1DI_2 et ABC. Montrer que

$$\frac{\rho}{r} \leq \frac{1}{2 + \sqrt{2}}.$$

3817. *Proposé par Tigran Hakobyan, Yerevan State University, Yerevan, Armenia.*

Soit $a, b \in \mathbb{N}$ avec $\gcd(a, b) = 1$. Soit $p_1 < p_2 < p_3 < \ldots$ l'ensemble des nombres premiers dans la progression $\{ak + b\}_{k=0}^{\infty}$. On considère

$$\alpha = 0.p_1p_2p_3\cdots,$$

où les chiffres des nombres premiers $p_1, p_2, p_3, \ldots$, placés côte à côte, forment les décimales de α. Montrer que α est irrationnel.

3818. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b, c trois nombres réels positifs tels que $abc = 1$. Montrer que

$$\frac{(\sqrt{a}+\sqrt{b})^4}{a+b} + \frac{(\sqrt{b}+\sqrt{c})^4}{b+c} + \frac{(\sqrt{c}+\sqrt{a})^4}{c+a} \geq 24.$$

3819. *Proposé par Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Espagne.*

Soit ABC un triangle, O le centre de son cercle circonscrit et I celui de son cercle inscrit. Soit ℓ une perpendiculaire quelconque à OI. Montrer que pour tout point P sur ℓ, situé à l'intérieur du triangle, la somme des distances de P aux côtés de ABC est constante.

3820. *Proposé par Michel Bataille, Rouen, France.*

Montrer que

$$\frac{2x}{\sinh(2\tanh x)} < (\cosh x)^2 < \frac{2x}{\sinh(2\tanh x)} + x\sinh(2x)$$

pour tout réel x non nul.

. . . .

3793. *Correction. Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, and c be positive real numbers such that

$$\sqrt{a}+\sqrt{b}+\sqrt{c} = 1007\sqrt{2}\ .$$

Find the minimum value of the expression

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\ .$$

3811. *Proposed by Jung In Lee, Seoul Science High School, Seoul, Republic of Korea.*

Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers a and b, $af(a+b)+bf(a)+b^2$ is a perfect square.

3812. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let $ABCD$ be a parallelogram and P be a point on side BC. Let K, L, and M be the centroids of triangles PAB, PAD and PCD, respectively. Prove that

$$[AKL] + [DLM] = [BKMC],$$

where $[\cdot]$ represents the area.

3813. *Proposed by Michel Bataille, Rouen, France.*

Find the smallest constant C such that the inequality

$$(a^7 + b^7 + c^7)^6 \leq C(a^6 + b^6 + c^6)^7$$

holds for all real numbers a, b, c such that $a + b + c = 0$.

3814. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Prove that for any number x in the closed interval $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$, there exists a point M in the plane of the square $ABCD$ such that

$$x = \frac{AM + MC}{BM + MD}.$$

3815. *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.*

Show that $x^x \leq x^2 - x + 1$ for all $0 \leq x \leq 1$.

3816. *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC be a right triangle with right angle at C, and let D be the foot of the altitude from C. Let I_1 and I_2 be the incentres of triangles CAD and CBD, respectively. Let ρ and r be the inradii of triangles I_1DI_2 and ABC, respectively. Prove that

$$\frac{\rho}{r} \leq \frac{1}{2 + \sqrt{2}}.$$

3817. *Proposed by Tigran Hakobyan, Yerevan State University, Yerevan, Armenia.*

Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Let $p_1 < p_2 < p_3 < \dots$ be the set of primes in the progression $\{ak + b\}_{k=0}^{\infty}$. Consider

$$\alpha = 0.p_1p_2p_3 \cdots ,$$

where the digits of the prime numbers $p_1, p_2, p_3, \dots$ placed side by side form the digits of α. Prove that α is irrational.

3818. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{(\sqrt{a} + \sqrt{b})^4}{a + b} + \frac{(\sqrt{b} + \sqrt{c})^4}{b + c} + \frac{(\sqrt{c} + \sqrt{a})^4}{c + a} \geq 24.$$

3819. *Proposed by Francisco Javier García Capitán, IES Álvarez Cubero, Priego de Córdoba, Spain.*

Let ABC be a triangle with circumcentre O and incentre I. Let ℓ be any line that is perpendicular to OI. Prove that for any point P on ℓ that is inside the triangle, the sum of the distances from P to the sides of ABC is constant.

3820. *Proposed by Michel Bataille, Rouen, France.*

Prove that

$$\frac{2x}{\sinh(2\tanh x)} < (\cosh x)^2 < \frac{2x}{\sinh(2\tanh x)} + x\sinh(2x)$$

for all nonzero real x.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3676. [2011 : 454, 456] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be the sides of a triangle with semiperimeter s, inradius r and circumradius R. Let r' and R' be the inradius and circumradius of a triangle with sides $\sqrt{a(s-a)}$, $\sqrt{b(s-b)}$, and $\sqrt{c(s-c)}$. Prove that

$$Rr' \geq R'r\,.$$

I. Solution by the proposer.

Let $a' = \sqrt{a(s-a)}$, $b' = \sqrt{b(s-b)}$, and $c' = \sqrt{c(s-c)}$. An easy calculation gives the following useful equality:

$$b'^2 + c'^2 - a'^2 = b(s-b) + c(s-c) - a(s-a) = 2(s-b)(s-c)\,. \qquad (1)$$

From (1), we have $a'^2 - b'^2 - c'^2 < 0 < 2b'c'$, and so consequently, $a' < b' + c'$. In a similar way, $b' < c' + a'$ and $c' < a' + b'$ and so triangles with sides a', b', c' do exist. Let $A'B'C'$ be such a triangle and let A', B', and C' be the angles opposite a', b', and c', respectively. Then using the law of cosines together with (1), the following equality is obtained:

$$\cos A' = \frac{2(s-b)(s-c)}{2\sqrt{bc(s-b)(s-c)}} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sin\frac{A}{2}\,. \qquad (2)$$

The last equality in (2) is because

$$\sin\frac{A}{2} = \frac{r}{\sqrt{r^2+(s-a)^2}} = \frac{rs}{\sqrt{r^2s^2+(s-a)^2s^2}}$$
$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c)+(s-a)^2s^2}} = \sqrt{\frac{(s-b)(s-c)}{bc}}\,. \qquad (3)$$

It is also useful to note that, because $0 < \frac{A}{2} < \frac{\pi}{2}$,

$$\cos\frac{A}{2} = \sqrt{1-\sin^2\frac{A}{2}} = \sqrt{\frac{s(s-a)}{bc}}\,. \qquad (4)$$

Similarly to equation (2), the following are true: $\cos B' = \sin\frac{B}{2}$ and $\cos C' = \sin\frac{C}{2}$. Next, recall the triangle formula:

$$\frac{r}{R} = \cos A + \cos B + \cos C - 1\,. \qquad (5)$$

Combining (2) and (5), yields the following equality:

$$\frac{r'}{R'} = \cos A' + \cos B' + \cos C' - 1 = \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} - 1\,. \qquad (6)$$

Combining (5) and (6), it follows that we need to prove

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq \cos A + \cos B + \cos C\,. \qquad (7)$$

But, this results from the obvious:

$$\left(2\sin\frac{A}{2}\right)\left(1-\cos\frac{B-C}{2}\right) + \left(2\sin\frac{B}{2}\right)\left(1-\cos\frac{C-A}{2}\right)$$
$$+\left(2\sin\frac{C}{2}\right)\left(1-\cos\frac{A-B}{2}\right) \geq 0\,. \qquad (8)$$

Using some trigonometric identities and triangle formulas, it can be shown that (8) can be rewritten as

$$\left(2\sin\frac{A}{2} - \cos B - \cos C\right) + \left(2\sin\frac{B}{2} - \cos C - \cos A\right)$$
$$+\left(2\sin\frac{C}{2} - \cos A - \cos B\right) \geq 0\,, \qquad (9)$$

which is just (7), and the desired inequality is proved. Note that one way to see that (8) and (9) are the same is to reduce them both to the same quantity in terms of just the side lengths a, b, c. This can be done by first applying the cosine of a difference rule, such as $\cos\frac{C-A}{2} = \cos\frac{C}{2}\cos\frac{A}{2} + \sin\frac{C}{2}\sin\frac{A}{2}$, and the double angle formula $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$, then applying equations (3) and (4), and finally substituting $s = \frac{1}{2}(a+b+c)$.

II. Solution by Arkady Alt, San Jose, CA, USA.

Let F be the area of the first triangle with sides a, b, c. Let s' and F' be the semiperimeter and area, respectively, of the second triangle with side lengths $a' = \sqrt{a(s-a)}$, $b' = \sqrt{b(s-b)}$, and $c' = \sqrt{c(s-c)}$. Recall the following formulas for a triangle with sides a, b, c and area F :

$$F = rs \tag{1}$$

$$abc = 4RF \tag{2}$$

$$F = \sqrt{s(s-a)(s-b)(s-c)} \tag{3}$$

This last formula, (3), is referred to as Heron's formula. Note that formulas (1)–(3) can be applied to the second triangle by placing a prime on each of the variables. Using (1) and (2), the goal is to show the following inequality:

$$Rr' \geq R'r \Leftrightarrow \frac{R}{r} \geq \frac{R'}{r'} \Leftrightarrow \frac{4RF}{rsF} \geq \frac{4R'F'}{sr'F'} \Leftrightarrow \frac{abc}{F^2} \geq \frac{a'b'c'}{sr'F'}\,. \tag{4}$$

By observing that

$$2s' = a' + b' + c' = \sqrt{2}\sum_{\text{cyclic}}\sqrt{\frac{a}{2}(s-a)} \leq \frac{\sqrt{2}}{2}\sum_{\text{cyclic}}\left(\frac{a}{2} + (s-a)\right) = \sqrt{2}s$$

and applying (1), the following inequality is established:

$$s \geq \sqrt{2}s' \Leftrightarrow \frac{a'b'c'}{\sqrt{2}s'r'F'} \geq \frac{a'b'c'}{sr'F'} \Leftrightarrow \frac{a'b'c'}{\sqrt{2}F'^2} \geq \frac{a'b'c'}{sr'F'}\,. \tag{5}$$

Because of (5), in order to obtain (4), it suffices to prove the following inequality:

$$\frac{abc}{F^2} \geq \frac{a'b'c'}{\sqrt{2}F'^2} \Leftrightarrow \frac{abc}{F^2} \geq \frac{\sqrt{abc}\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{2}F'^2} \Leftrightarrow \sqrt{2}F'^2\sqrt{sabc} \geq F^3\,, \tag{6}$$

where Heron's formula was used to obtain the second equivalence in (6). Next, using Heron's formula for the triangle with sides a', b', c', and substituting $s' = \frac{1}{2}(a' + b' + c')$, observe that the following equality is true:

$$16F'^2 = 4\sum_{\text{cyclic}} a'^2b'^2 - \left(\sum_{\text{cyclic}} a'^2\right)^2 = 4\sum_{\text{cyclic}} ab(s-a)(s-b) - \left(\sum_{\text{cyclic}} a(s-a)\right)^2\,. \tag{7}$$

Therefore, from (7) and knowing that $s = \frac{1}{2}(a+b+c)$, the following equality is established:

$$16F'^2 = 4\left((ab+bc+ca)s^2 - s^4 - sabc\right) = 4\left((ab+bc+ca)s^2 - s^4 - 4Rrs^2\right)\,, \tag{8}$$

where (1) and (2) are used to obtain the last equality in (8). Next, using (1)–(3) and $s = \frac{1}{2}(a+b+c)$, it can be shown that $ab+bc+ca = s^2+4Rr+r^2$. Therefore, from (8), the following is obtained:

$$16F'^2 = 4\left((s^2 + 4Rr + r^2)s^2 - s^4 - 4Rrs^2\right) = 4F^2\,. \tag{9}$$

Finally, Euler's formula for the distance between the incenter and circumcenter, combined with formulas (1), (2) and (9), yields the following inequality:

$$\begin{aligned} R \geq 2r \Leftrightarrow 4Rrs^2 \geq 8r^2s^2 \quad & \Leftrightarrow sabc \geq 8F^2 \\ \Leftrightarrow \sqrt{sabc} \geq 2\sqrt{2}F \quad & \Leftrightarrow F^2\sqrt{sabc} \geq 2\sqrt{2}F^3 \\ \Leftrightarrow 4F'^2\sqrt{sabc} \geq 2\sqrt{2}F^3 \quad & \Leftrightarrow \sqrt{2}F'^2\sqrt{sabc} \geq F^3\,. \qquad (10) \end{aligned}$$

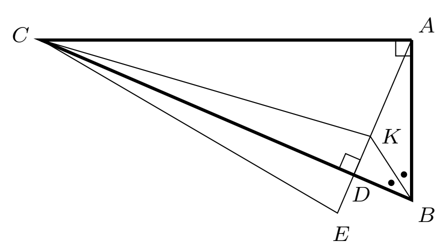
But, this is just (6), and the desired inequality is proved.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MARIAN DINCĂ, Bucharest, Romania; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and TITU ZVONARU, Comăneşti, Romania.

3711. [2012 : 63, 65] *Proposed by Mehmet Sahin, Ankara, Turkey.*

Let ABC be a right triangle with $\angle A = 90^\circ$. Let AD be an altitude, and let the angle bisector of $\angle B$ meet AD in K. If $\angle ACK = 2\angle DCK$ then prove that $KC = 2AD$.

Solution by Dag Jonsson, Uppsala, Sweden.



Since $\triangle ADB$ is similar to $\triangle CDA$ (and similar to $\triangle CAB$),

$$\frac{AD}{AC} = \frac{BD}{BA}\,. \qquad (1)$$

Let E be the point on the extension of AD beyond D, such that $DE = KD$. Then the triangles KDC and EDC are congruent, giving $EC = KC$.
Applying the bisector theorem to triangles ABD and ACE, and using equation (1), we obtain

$$2\frac{AD}{AC} = 2\frac{BD}{BA} = 2\frac{KD}{KA} = \frac{KE}{KA} = \frac{EC}{AC} = \frac{KC}{AC}.$$

Thus, $KC = 2AD$.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, *Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; GEORGES MELKI, Fanar, Lebanon; JOEL SCHLOSBERG, Bayside, NY, USA; MIHAÏ STOËNESCU, Bischwiller, France; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.*

3712. [2012 : 63, 65] *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

Prove that for any positive real numbers a, b, c

$$\sqrt{\frac{a(a^2+bc)}{b+c}}+\sqrt{\frac{b(b^2+ca)}{c+a}}+\sqrt{\frac{c(c^2+ab)}{a+b}} \geq a+b+c.$$

I. Composite of similar solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, NRW, Germany; John G. Heuver, Grande Prairie, AB; and Salem Malikić, student, Simon Fraser University, Burnaby, BC.

By symmetry, we may assume that $a \geq b \geq c > 0$. We prove that

$$\sqrt{\frac{c\left(c^2+ab\right)}{a+b}} \geq c \tag{1}$$

and that

$$\sqrt{\frac{a\left(a^2+bc\right)}{b+c}}+\sqrt{\frac{b\left(b^2+ca\right)}{c+a}} \geq a+b, \tag{2}$$

which together prove the claim.

Inequality (1) is equivalent in succession to,

$$\frac{c\left(c^2+ab\right)}{a+b} \geq c^2,$$
$$c^2+ab \geq c\,(a+b)\,,$$
$$(a-c)\,(b-c) \geq 0,$$

which is true by our assumption $a \geq b \geq c > 0$.

Inequality (2) is equivalent to

$$\frac{a\left(a^2+bc\right)}{b+c}+\frac{b\left(b^2+ca\right)}{c+a}+2\sqrt{\frac{a\left(a^2+bc\right)}{b+c}\cdot\frac{b\left(b^2+ca\right)}{c+a}} \geq a^2+b^2+2ab. \tag{3}$$

To prove (3), it suffices to prove

$$\frac{a\left(a^2+bc\right)}{b+c}+\frac{b\left(b^2+ca\right)}{c+a} \geq a^2+b^2 \tag{4}$$

and

$$\frac{a\left(a^2+bc\right)}{b+c}\cdot\frac{b\left(b^2+ca\right)}{c+a} \geq a^2b^2. \tag{5}$$

Some algebra shows that (4) is equivalent to $(a-b)^2\left(a^2+ab+b^2-c^2\right)\geq 0$ and that (5) is equivalent to $c\,(a-b)^2\,(a+b)\geq 0$, which are both true by our assumption $a\geq b\geq c>0$; hence the proof is complete.

II. Solution by the proposer, expanded slightly by the editor.

By Hölder's inequality,

$$\left(\sum_{\text{cyclic}}\sqrt{\frac{a\left(a^2+bc\right)}{b+c}}\right)^2\cdot\sum_{\text{cyclic}}\frac{a^2\,(b+c)}{a^2+bc}\geq\left(\sum_{\text{cyclic}}a\right)^3,$$

so it suffices to show that

$$\sum_{\text{cyclic}}\frac{a^2\,(b+c)}{a^2+bc}\leq\sum_{\text{cyclic}}a.$$

We have

$$\begin{aligned}\sum_{\text{cyclic}}\frac{a^2\,(b+c)}{a^2+bc}&=\sum_{\text{cyclic}}\frac{a^2\,(b+c)^2}{\left(a^2+bc\right)(b+c)}\\&=\sum_{\text{cyclic}}\frac{a^2\,(b+c)^2}{b\left(a^2+c^2\right)+c\left(a^2+b^2\right)}.\end{aligned}$$

By the Schwarz inequality in the form

$$\frac{(a_1+a_2)^2}{b_1+b_2}\leq\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2},$$

we have

$$\frac{a^2\,(b+c)^2}{b\left(a^2+c^2\right)+c\left(a^2+b^2\right)}\leq\frac{a^2b^2}{b\left(a^2+c^2\right)}+\frac{a^2c^2}{c\left(a^2+b^2\right)}=\frac{a^2b}{a^2+c^2}+\frac{a^2c}{a^2+b^2},$$

so that

$$\begin{aligned}\sum_{\text{cyclic}}\frac{a^2\,(b+c)}{a^2+bc}&\leq\sum_{\text{cyclic}}\left(\frac{a^2b}{a^2+c^2}+\frac{a^2c}{a^2+b^2}\right)\\&=\sum_{\text{cyclic}}\left(\frac{a^2b}{a^2+c^2}+\frac{c^2b}{a^2+c^2}\right)\\&=\sum_{\text{cyclic}}b.\end{aligned}$$

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece (2 solutions); MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ÁNGEL PLAZA, *University of Las Palmas de Gran Canaria, Spain; YUNUS TUNCBILEK, Ataturk High School of Science'14, Istanbul, Turkey; and TITU ZVONARU, Comăneşti, Romania. One incorrect solution was received.*

Several solvers used a generalized Schur inequality, with Tuncbilek explicitly citing a theorem from Pham Kim Hung, Secrets in Inequalities, *Vol 2, GIL Publishing House, 2008. Other solvers used variants of the methods of the two featured solutions. Geupel and Heuver referenced Joe Howard's solution to problem 3554 [2011 : 327] as inspiration for their solutions. Lau used the convexity of the function $1/\sqrt{x}$.*

3713. [2012 : 63, 65] *Proposed by D. M. Bătineţu, Giurgiu, Bucharest and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Compute

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!c_n}}-\frac{n^2}{\sqrt[n]{(2n-1)!!e_n}}\right),$$

where $e_n=\left(1+\dfrac{1}{n}\right)^n$ and $c_n=-\ln n+\displaystyle\sum_{k=1}^{n}\frac{1}{k}$, for any positive integer n.

I. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

The answer is $(e/2)(2-\ln\gamma)$, where $\gamma=\lim_{n\to\infty}c_n=0.577215665\ldots$ is Euler's constant. Recall that $(2n-1)!!=(2n)!2^{-n}(n!)^{-1}=1\cdot3\cdot5\cdots(2n-1)$, the product of the first n odd numbers.

We first establish the following result:

Let $\{a_n\}$ and $\{b_n\}$ be real sequences that satisfy

(a) $\displaystyle\lim_{n\to\infty}\frac{a_n}{n}=\alpha\in(0,\infty)$,

(b) $\displaystyle\lim_{n\to\infty}\frac{b_{n+1}}{a_n}=1$,

(c) $\displaystyle\lim_{n\to\infty}\left(\frac{b_{n+1}}{a_n}\right)^n=\beta\in(0,\infty)$.

Then

$$\lim_{n\to\infty}(b_{n+1}-a_n)=\alpha\ln\beta.$$

Observe that, when $b_{n+1}\neq a_n$,

$$(b_{n+1}-a_n)\left(\frac{a_n}{b_{n+1}-a_n}\right)\ln\left(1+\frac{b_{n+1}-a_n}{a_n}\right)=\frac{a_n}{n}\cdot\ln\left(\frac{b_{n+1}}{a_n}\right)^n.$$

Since $\displaystyle\lim_{n\to\infty}(b_{n+1}-a_n)a_n^{-1}=0$ and $\displaystyle\lim_{t\to0}t^{-1}\ln(1+t)=1$, we deduce that $\displaystyle\lim_{n\to\infty}(b_{n+1}-a_n)=\alpha\ln\beta$. (If $\beta\neq1$, then b_{n+1} is eventually distinct from a_n, while, if $\beta=1$, a slight modification to the argument leads to the same result.)

We now apply this result to

$$b_{n+1}=\sqrt[n+1]{\frac{(n+1)^{2n+2}}{(2n+1)!!c_n}}$$

and

$$a_n = \sqrt[n]{\frac{n^{2n}}{(2n-1)!!e_n}}$$

and show that

$$\lim_{n\to\infty} \frac{a_n}{n} = \frac{e}{2};$$

$$\lim_{n\to\infty} \frac{b_{n+1}}{a_n} = 1;$$

and

$$\lim_{n\to\infty} \left(\frac{b_{n+1}}{a_n}\right)^n = \frac{e^2}{\gamma},$$

from which it will follow that $\lim_{n\to\infty}(b_{n+1} - a_n) = \frac{e}{2}(2 - \ln\gamma)$.

First, applying the equal values given by the ratio and root tests, we obtain

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2n-1)!!e_n}} = \lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{(2n+1)!!e_{n+1}}\right)\left(\frac{(2n-1)!!e_n}{n^n}\right)$$
$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(\frac{n+1}{2n+1}\right)\left(\frac{e_n}{e_{n+1}}\right) = \frac{e}{2}.$$

Secondly,

$$\lim_{n\to\infty} \frac{b_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{a_n}\cdot\frac{n+1}{n}\cdot\frac{b_{n+1}}{n+1} = \frac{2}{e}\lim_{n\to\infty}\frac{b_{n+1}}{n+1}$$
$$= \frac{2}{e}\lim_{n\to\infty} \sqrt[n+1]{\frac{(n+1)^{n+1}}{(2n+1)!!c_n}}$$
$$= \frac{2}{e}\lim_{n\to\infty} \left(\frac{(n+1)^{n+1}}{(2n+1)!!c_n}\right)\left(\frac{(2n-1)!!c_{n-1}}{n^n}\right)$$
$$= \frac{2}{e}\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(\frac{n+1}{2n+1}\right)\left(\frac{c_{n-1}}{c_n}\right) = \frac{2}{e}\cdot e\cdot\frac{1}{2} = 1.$$

Finally,

$$n\ln\frac{b_{n+1}}{a_n} = n\ln\left(\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!c_n}}\right)\left(\frac{\sqrt[n]{(2n-1)!!e_n}}{n^2}\right)$$
$$= 2\ln\left(1+\frac{1}{n}\right)^n + \ln(2n-1)!! + \ln e_n - \frac{n\ln(2n+1)!!}{n+1} - \frac{n\ln c_n}{n+1}$$
$$= 3\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + n\left(\frac{1}{n}\sum_{k=1}^{n}\ln(2k-1) - \frac{1}{n+1}\sum_{k=1}^{n}\ln(2k+1)\right)$$
$$= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1}$$
$$+ n\left(\frac{1}{n}\sum_{k=1}^{n-1}\ln(2k+1) - \frac{1}{n+1}\sum_{k=1}^{n-1}\ln(2k+1) - \frac{\ln(2n+1)}{n+1}\right)$$

$$\begin{aligned}
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + n\left(\frac{1}{n(n+1)}\sum_{k=1}^{n-1}\ln(2k+1) - \frac{\ln(2n+1)}{n+1}\right)\\
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + \frac{\ln(2n-1)!!}{n+1} - \frac{\ln(2n+1)^n}{n+1}\\
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + \frac{1}{n+1}\ln\frac{(2n-1)!!}{(2n+1)^n}\\
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + \frac{1}{n+1}\ln\left(\frac{(2n)!}{(2n+1)^{2n}}\cdot\frac{(2n+1)^n}{2^n\cdot n!}\right)\\
&= 3\ln\left(1+\frac{1}{n}\right)^n - \frac{n\ln c_n}{n+1} + \left(\frac{2n+1}{n+1}\right)\left[\frac{1}{2n+1}\ln\left(\frac{(2n+1)!}{(2n+1)^{2n+1}}\right)\right]\\
&\qquad + \left(\frac{n}{n+1}\right)\left[\frac{1}{n}\ln\left(\frac{(2n+1)^n}{2^n\cdot n!}\right)\right]
\end{aligned}$$

We need to apply Stirling's formula:

$$\ln(n!) = n\ln n - n + \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi) + O\left(\frac{1}{n}\right),$$

so that

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{n!}{n^n}\right) = \lim_{n\to\infty}\frac{1}{n}[\ln n! - n\ln n] = -1.$$

Note also that

$$\frac{n^n}{n!} < \frac{(2n+1)^n}{2^n\cdot n!} < \frac{(n+1)^{n+1}}{(n+1)!}.$$

Therefore

$$\lim_{n\to\infty} n\ln\frac{b_{n+1}}{a_n} = 3 - \ln\gamma - 2 + 1 = 2 - \ln\gamma.$$

The solution is complete.

II. Solution by Michel Bataille, Rouen, France.

We shall show that the required limit is $e(1-\ln\sqrt{\gamma})$. We have that

$$\begin{aligned}
\frac{n^2}{\sqrt[n]{(2n-1)!!e_n}} &= 2n^2\left(1+\frac{1}{n}\right)^{-1}\exp\frac{1}{n}(\ln(n)! - \ln(2n)!)\\
&= 2n^2\left(1-\frac{1}{n}+o(1/n)\right)\\
&\qquad\times\exp\left(\ln n - 1 + \frac{\ln n}{2n} - 2\ln 2 - 2\ln n + 2 - \frac{\ln 2}{2n} - \frac{\ln n}{2n} + o(1/n)\right)\\
&= 2n^2\left(1-\frac{1}{n}+o(1/n)\right)\left(\frac{e}{4n}\right)\exp\left(-\frac{\ln 2}{2n}+o(1/n)\right)\\
&= \frac{en}{2}\left(1-\frac{1}{n}+o(1/n)\right)\left(1-\frac{\ln 2}{2n}+o(1/n)\right)\\
&= \frac{en}{2} - \frac{e}{2}\left(1+\frac{\ln 2}{2}\right) + o(1).
\end{aligned}$$

Since $c_n = \gamma + o(1)$, we have also that

$$\frac{(n+1)^2}{\sqrt[n+1]{(2n+1)!!c_n}} = \exp\left(\frac{-1}{n+1}\ln c_n\right) 2(n+1)^2 \left(\frac{e}{4(n+1)}\right) \exp\left(-\frac{\ln 2}{2(n+1)} + o(1/n)\right)$$
$$= \exp\left[\left(\frac{-1}{n} + o(1/n)\right)\ln(\gamma + o(1))\right]$$
$$\times \left(\frac{e(n+1)}{2}\right)\left(1 - \frac{\ln 2}{2(n+1)} + o(1/n)\right)$$
$$= \left(1 - \frac{\ln\gamma}{n} + o(1/n)\right)\left(\frac{en}{2} + \frac{e}{2} - \frac{e\ln 2}{4} + o(1)\right)$$
$$= \frac{en}{2} + \frac{e}{2}\left(1 - \frac{\ln 2}{2} - \ln\gamma\right) + o(1).$$

The required difference is therefore $e - \frac{e}{2}\ln\gamma + o(1)$, and we obtain the result.

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANASTASIOS KOTRONIS, Athens, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposers.

3714. [2012 : 64, 66] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Given a triangle ABC define points B_1 on side AB and C_1 on AC so that $B_1C_1 \parallel BC$; similarly take C_2 and A_2 on sides BC and BA with $C_2A_2 \parallel CA$, and A_3, B_3 on CA, CB with $A_3B_3 \parallel AB$. Furthermore, denote A_i', B_i', C_i' the projections of A_i, B_i, C_i onto the corresponding parallel sides of the given triangle (to form three rectangles such as $B_1B_1'C_1'C_1$).

(a) Prove that if the ratios of the areas of each defined triangle to that of its adjacent rectangle are equal, namely

$$\frac{[AB_1C_1]}{[B_1B_1'C_1'C_1]} = \frac{[BC_2A_2]}{[C_2C_2'A_2'A_2]} = \frac{[CA_3B_3]}{[A_3A_3'B_3'B_3]},$$

then the inradii of those three triangles are also equal.

(b) Determine the ratio of the inradius of the triangle formed by the lines C_1B_1, A_2C_2, B_3A_3 to the inradius of ΔABC.

Composite of solutions by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and by Titu Zvonaru, Comănești, Romania.

Let a, b, c be the lengths of the sides of ΔABC, and h be the altitude from A. We set

$$x = \frac{AB_1}{c}, \quad y = \frac{BC_2}{a}, \quad z = \frac{CA_3}{b}.$$

Because triangles are similar if their corresponding sides are parallel, we obtain $B_1C_1 = ax$ and $[AB_1C_1] = x^2[ABC]$. Furthermore, we have $\frac{BB_1}{BA} = \frac{B_1B_1'}{h}$, which is equivalent to $\frac{(1-x)c}{c} = \frac{B_1B_1'}{h}$, or

$$BB_1' = (1-x)h.$$

It follows that

$$[B_1B_1'C_1'C_1] = B_1C_1 \cdot B_1B_1' = ax \cdot (1-x)h = x(1-x) \cdot 2[ABC],$$

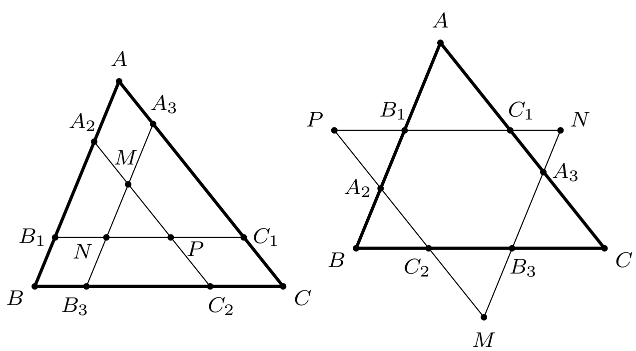
and finally,

$$\frac{[AB_1C_1]}{[B_1B_1'C_1'C_1]} = \frac{x}{2(1-x)}.$$

Solution to (a). Using similar arguments for y and z, we see that the given conditions are equivalent to

$$\frac{x}{1-x} = \frac{y}{1-y} = \frac{z}{1-z};$$

consequently, $x = y = z$, so that the triangles AB_1C_1, BC_2A_2, CA_3B_3 are congruent and, therefore, the inradii of these three triangles are equal; indeed, they are all equal to xr, where r is the inradius of ΔABC.



Solution to (b). The accompanying figures indicate that if we use directed distances, then special cases will not be required. Let MNP be the triangle formed by the lines C_1B_1, A_2C_2, B_3A_3. Using the similar triangles of part (a) we have $C_1A = bx$, $CA_3 = bz$, and, therefore, $A_3C_1 = b(1-x-z)$. From $\Delta AB_1C_1 \sim \Delta A_3NC_1$ we have $\dfrac{A_3C_1}{C_1N} = \dfrac{AC_1}{C_1B_1}$, which is equivalent to $\dfrac{b(1-x-z)}{C_1N} = \dfrac{bx}{ax}$, so that

$$C_1N = a(1-x-z).$$

Similarly,

$$PB_1 = a(1-x-y).$$

It follows that

$$\begin{aligned} PN &= PB_1 + B_1C_1 + C_1N \\ &= a(1-x-y) + ax + a(1-x-z) = a(2-x-y-z). \end{aligned}$$

Because $\Delta MNP \sim \Delta ABC$, we conclude that $|2-x-y-z|r$ is the inradius of ΔMNP, where r is the inradius of ΔABC. More illuminating, the ratio of the dilatation taking the segment BC to NP is $x+y+z-2$, which is positive when the ratios x, y, z are relatively large (as in the figure on the left), and negative when they are relatively small (as on the right).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

In the special case of part (a) when $x = y = z$, the ratio of the inradii of part (b) is $3x - 2$. In terms of the common ratio $\frac{[AB_1C_1]}{[B_1B_1'C_1'C_1]} = \frac{x}{2(1-x)}$, call it t, the ratio of the inradii would be $\frac{2t-2}{2t+1}$.

Swylan observed that the results continue to hold if instead of restricting the new points A_i, B_i, C_i to the sides of the original triangle, we allow them to lie on the extensions of the sides, but only if we use signed areas in part (a). For a counterexample take $x = z = \frac{2}{3}$ and $y = 2$. The resulting area ratios are equal in magnitude (but not in sign), yet one inradius is three times the size of the other two.

3715. [2012 : 64, 66] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let h_a, h_b, h_c be the altitudes, r_a, r_b, r_c the exradii, r the inradius and R the circumradius of a triangle. Prove that

$$\frac{h_a^2}{r_a} + \frac{h_b^2}{r_b} + \frac{h_c^2}{r_c} \geq 4r\left(2 - \frac{r}{R}\right)^2.$$

Solution by Kee-Wai Lau, Hong Kong, China.

Denoting the semiperimeter by $s = \frac{1}{2}(a+b+c)$, we know that $r_a = \frac{rs}{s-a}$ and $h_a = \frac{bc}{2R}$, hence

$$\frac{h_a^2}{r_a} = \frac{b^2c^2(s-a)}{4srR^2}.$$

Similarly,

$$\frac{h_b^2}{r_b} = \frac{c^2a^2(s-b)}{4srR^2} \quad \text{and} \quad \frac{h_c^2}{r_c} = \frac{a^2b^2(s-c)}{4srR^2}.$$

Therefore, the inequality of the problem is equivalent to

$$s\left(a^2b^2 + b^2c^2 + c^2a^2\right) - abc(ab + bc + ca) - 16sr^2(2R - r)^2 \geq 0. \qquad (1)$$

We modify (1) by applying the identities

$$abc = 4srR, \quad ab + bc + ca = s^2 + r^2 + 4rR,$$

and

$$\begin{aligned} a^2b^2 + b^2c^2 + c^2a^2 &= (ab + bc + ca)^2 - 2abc(a + b + c) \\ &= s^4 + r^4 + 16R^2r^2 + 2r^2s^2 + 8Rr^3 - 8s^2Rr \end{aligned}$$

to obtain

$$s^4 + 2r(r - 6R)s^2 - 64r^2R^2 + 68Rr^3 - 15r^4 \geq 0. \qquad (2)$$

The left side of (2) equals $\left(s^2+4Rr-3r^2\right)\left(s^2-16Rr+5r^2\right)$. By Euler's inequality $R \geq 2r$, whence $s^2+4Rr-3r^2>0$. Finally, according to J.C.H. Gerretsen (see Formula 5.8 in O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969), $s^2-16Rr+5r^2 \geq 0$, with equality if and only if the given triangle is equilateral. Thus (2) holds, and we have proved that the original inequality is strict unless the triangle is equilateral.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; EDMUND SWYLAN, Riga, Latvia; DANIEL VĂCARU, Piteşti, Romania; TITU ZVONARU, Cománeşti, Romania; and the proposer.

3716. [2012 : 64, 66] *Proposed by Michel Bataille, Rouen, France.*

Given positive integers a, u_1, u_2 let u_n be defined by the recursion

$$u_{n+2} = 2au_{n+1} + u_n,\ n \in \mathbb{N}.$$

Show that there exists a positive real number r such that

$$u_{n+2^{k+1}} = \lfloor 1 + r^{2^k} \rfloor \cdot u_{n+2^k} - u_n$$

for all positive integers n, k.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The linear recursion has the characteristic polynomial $x^2-2ax-1$ with the distinct roots $\lambda = a+\sqrt{a^2+1}$ and $-\lambda^{-1}$. Hence, there are real constants b, c such that $u_n = b\lambda^n + c(-\lambda)^{-n}$. For positive integers n, k let the number $q_{n,k}$ be defined by

$$u_{n+2^{k+1}} = (1+q_{n,k}) \cdot u_{n+2^k} - u_n.$$

Then

$$q_{n,k} = \frac{u_n - u_{n+2^k} + u_{n+2^{k+1}}}{u_{n+2^k}} = \frac{\left(\lambda^{2^{k+1}} - \lambda^{2^k} + 1\right)\left(b\lambda^{2n+2^{k+1}} + (-1)^n c\right)}{\lambda^{2^k}\left(b\lambda^{2n+2^{k+1}} + (-1)^n c\right)}$$
$$= \lambda^{2^k} - 1 + \lambda^{-2^k}.$$

Note that $q_{n,k}$ is independent of n. Therefore we can write q_k instead of $q_{n,k}$.

Consider the closed intervals $C_k = \left[q_k^{1/2^{k-1}},\ (q_k+1)^{1/2^{k-1}}\right]$, $\quad k = 1, 2, \ldots$. We have $q_1 = 4a^2+1$ and

$$q_{k+1} = \lambda^{2^{k+1}} - 1 + \lambda^{-2^{k+1}} = \left(\lambda^{2^k} + \lambda^{-2^k}\right)^2 - 3 = (q_k+1)^2 - 3.$$

Hence, q_k is a positive integer for each $k > 0$. We obtain

$$q_k^2 < q_k^2 + 2(q_k - 1) = (q_k+1)^2 - 3 = q_{k+1},$$

that is

$$q_1 < q_2^{1/2} < q_3^{1/4} < q_4^{1/8} < \cdots .$$

Moreover, $q_{k+1} + 1 = (q_k + 1)^2 - 2 < (q_k + 1)^2$. Whence,

$$\cdots < (q_4 + 1)^{1/8} < (q_3 + 1)^{1/4} < (q_2 + 1)^{1/2} < q_1 + 1.$$

Therefore, $C_1 \supset C_2 \supset C_3 \supset C_4 \supset \cdots$. By Cantor's intersection theorem, there is a positive real number r such that

$$r^2 \in \bigcap_{k=1}^{\infty} C_k.$$

Thus, $q_k \leq r^{2^k} \leq q_k + 1$. Here the right inequality is strict because $r^{2^{k+1}} \leq q_{k+1} + 1 < (q_k + 1)^2$ implies $r^{2^k} < q_k + 1$. We conclude

$$q_k = \lfloor r^{2^k} \rfloor.$$

This completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

3717. [2012 : 64, 66] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Find all real solutions $(x_1, x_2, \ldots, x_n)$ of the system of equations

$$\begin{aligned} x_1 &= \exp\left[\sin\left(x_2 - \sqrt{1 - \ln^2 x_1}\right)\right] \\ x_2 &= \exp\left[\sin\left(x_3 - \sqrt{1 - \ln^2 x_2}\right)\right] \\ &\vdots \\ x_n &= \exp\left[\sin\left(x_1 - \sqrt{1 - \ln^2 x_n}\right)\right]. \end{aligned}$$

Comment by the Editor: There were two submitted solutions in addition to the one given by the proposer. These solutions were quite similar and all claimed that the only solution is $x_1 = x_2 = ... = x_n = 1$. However, they all made the following common mistake: after changing variables in the original equations to some equivalent ones, they took arcsin *of both sides of the resulting equations, ignoring the fact that $y = \sin(x)$ is not equivalent to $x = \arcsin(y)$ unless it can be shown that x is in the interval $[-\pi/2, \pi/2]$ (and it is not in the present case). Actually an in-depth analysis of the implicit function $x = \sin(e^y - \sqrt{1 - x^2})$ shows that it has exactly two fixed points which are both solutions to the given system which therefore must have at least two different solutions. Therefore, this problem remains open.*

3718. [2012 : 65, 67] *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b and c be real numbers such that $a > b > c$ and $b + c = 503$.

(i) Find the minimum value of the expression

$$A = \frac{a^2}{a-b} + \frac{b^2}{b-c}.$$

(ii) Determine values of a, b, c for which A attains its minimum value.

I. Solution by Marian Dincă, Bucharest, Romania.

Let $x = a - b$ and $y = b - c$. Using the Arithmetic-Geometric Means Inequality, we obtain that

$$\begin{aligned} A &= \frac{(b+x)^2}{x} + \frac{(c+y)^2}{y} = \left(\frac{b^2}{x} + x\right) + \left(\frac{c^2}{y} + y\right) + 2(b+c) \\ &\geq 2b + 2c + 2(b+c) = 4(b+c) = 4 \cdot 503 = 2012, \end{aligned}$$

with equality if and only if $b = x$ and $c = y$, which is equivalent to $a = 2b = 4c$. This in turn requires that

$$(a, b, c) = \left(\frac{2012}{3}, \frac{1006}{3}, \frac{503}{3}\right).$$

II. Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Salem Malikić, student, Simon Fraser University, Burnaby, BC (independently).

$$\begin{aligned} A &= \left(\frac{a}{\sqrt{a-b}} - 2\sqrt{a-b}\right)^2 + \left(\frac{b}{\sqrt{b-c}} - 2\sqrt{b-c}\right)^2 + 4(b+c) \\ &\geq 4(b+c) = 2012, \end{aligned}$$

with equality if and only if $a = 2(a-b)$ and $b = 2(b-c)$, *i.e.* $(a, b, c) = (2012/3, 1006/3, 503/3)$.

III. Solution by Arkady Alt, San Jose, CA, USA.

Since for any positive reals x, y, and p, $p^2x^2/y \geq 2px - y$ with equality if and only if $px = y$, we have that

$$\begin{aligned} p^2 A &= \frac{p^2a^2}{a-b} + \frac{p^2b^2}{b-c} \\ &\geq 2pa - (a-b) + 2pb - (b-c) = (2p-1)a + 2pb + c. \end{aligned}$$

Taking $p = \frac{1}{2}$ yields that $A \geq 2012$ with equality if and only if $a = 2(a-b)$, $b = 2(b-c)$. With $b + c = 503$ this gives the values of a, b, c recorded above.

IV. Solution by Kee-Wai Lau, Hong Kong, China.

Since $b > c$ and $b + c = 503$, then $2b > 503$. We have that

$$A = \frac{a^2}{a-b} + \frac{b^2}{2b-503} = \frac{(a-2b)^2}{a-b} + \frac{(3b-1006)^2}{2b-503} + 2012.$$

Therefore A attains its minimum value if and only if $3b = 1006$, $a = 2b$.

Also solved by MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; JOEL SCHLOSBERG, Bayside, NY, USA; DANIEL VĂCARU, Piteşti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer.

Some solvers resorted to calculus, one using Lagrange Multipliers.

3719. [2012 : 65, 67] *Proposed by Johan Gunardi, student, SMPK 4 BPK PENABUR, Jakarta, Indonesia.*

Prove that for any positive real numbers a, b, c,

$$\sqrt{\frac{a(a^2+bc)}{b+c}} + \sqrt{\frac{b(b^2+ca)}{c+a}} + \sqrt{\frac{c(c^2+ab)}{a+b}} \geq a+b+c.$$

Composite of essentially the same solution by Radouan Boukharfane, Polytechnique de Montréal, PQ; Oliver Geupel, Brühl, NRW, Germany; John G. Heuver, Grande Prairie, AB; and Itachi Uchiha, Hong Kong, China.

By Hölder's Inequality, we have

$$\begin{aligned}\left(\sum_{\text{cyclic}} \frac{a}{\sqrt{4b^2+bc+4c^2}}\right)^{\frac{2}{3}} & \left(\sum_{\text{cyclic}} a(4b^2+bc+4c^2)\right)^{\frac{1}{3}} \\ &\geq \sum_{\text{cyclic}} \left(\frac{a^{\frac{2}{3}}}{(4b^2+bc+4c^2)^{\frac{1}{3}}}\right)\left(a^{\frac{1}{3}}(4b^2+bc+4c^2)^{\frac{1}{3}}\right) \\ &= a+b+c\end{aligned}$$

so

$$\left(\sum_{\text{cyclic}} \frac{a}{\sqrt{4b^2+bc+4c^2}}\right)^2 \left(\sum_{\text{cyclic}} a(4b^2+bc+4c^2)\right) \geq (a+b+c)^3. \qquad (1)$$

Next, by Schur's Inequality, we have

$$\left(\sum_{\text{cyclic}} a^3\right) + 3abc \geq \sum_{\text{cyclic}} (a^2b+ab^2).$$

Hence,

$$(a+b+c)^3 = \left(\sum_{\text{cyclic}} a^3\right) + 3abc + 3\sum_{\text{cyclic}}(a^2b+ab^2) + 3abc$$
$$\geq 4\sum_{\text{cyclic}} ab(a+b) + 3abc = \sum_{\text{cyclic}} a(4b^2+bc+4c^2)\,. \qquad (2)$$

From (1) and (2) the result follows.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; HAOHAO WANG and YANPING XIA, Southeast Missouri State University, Cape Girardeau, MO, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer. There was also one incorrect solution. Almost all the submitted solutions used two of the following: convexity and Jensen's Inequality; Cauchy-Schwarz's Inequality; Hölder's Inequality; and Schur's Inequality.

3720. [2012 : 65, 67] *Proposed by Michel Bataille, Rouen, France.*

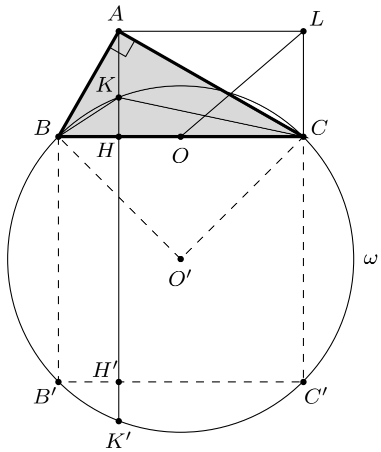
Let ABC be a triangle with $\angle BAC = 90^\circ$, O be the midpoint of BC and H be the foot of the altitude from A. Let K, on segment AH, be such that $\angle BKC = 135^\circ$ and L be such that $AHCL$ is a rectangle. Show that $OL = OB + KH$.

Composite of solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Let $a = BC$ and $x = KH$. Then we have $OB + KH = \frac{a}{2} + x$ and $OL = \sqrt{AH^2 + \left(\frac{a}{2}\right)^2}$. It is sufficient to prove that $OL^2 = (OB+KH)^2$; that is, $AH^2 + \left(\frac{a}{2}\right)^2 = \left(\frac{a}{2}+x\right)^2$, or

$$AH^2 = ax + x^2.$$

Let us denote by ω the circumcircle of triangle BKC and by O' its centre. Because $\angle BKC = 135^\circ$, the angle at O' subtended by the chord BC must be 90°, whence BC is one side of a square $BCC'B'$ inscribed in ω. Denote by K' and H' the points where the line AK meets ω (again) and $B'C'$, respectively. By symmetry, $KH' = HK' = a + x$. Because the chords KK' and BC intersect at H, we have



$$BH \cdot HC = KH \cdot HK' = x \cdot (a + x). \tag{1}$$

Because AH is the altitude to the hypotenuse of the right triangle ABC, we also have

$$BH \cdot HC = AH^2. \tag{2}$$

From (1) and (2) we have $AH^2 = x \cdot (a + x)$, as desired.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; ROY BARBARA, Lebanese University, Fanar, Lebanon; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; DAG JONSSON, Uppsala, Sweden; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.