SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The editor would like to acknowledge the solution to problem 3654 by PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. The editor apologizes sincerely for the oversight.

3674★. [2011 : 390, 392; 2012 : 347] *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let I denote the centre of the inscribed sphere of a tetrahedron $ABCD$ and let A_1, B_1, C_1, D_1 denote their symmetric points of point I about planes BCD, ACD, ABD, ABC respectively. Must the four lines AA_1, BB_1, CC_1, DD_1 be concurrent?

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.

No. We show that in a rectangular coordinate system with $A = (a, 0, 0)$, $B = (0, b, 0)$, $C = (0, 0, c)$ and $D = (0, 0, 0)$ (a, b, c positive), the three lines AA_1, BB_1, CC_1 are concurrent only if $a = b = c$.

Let r be the radius of the inscribed sphere. Its center I is the point (r, r, r). The symmetric points A', B', C' are $(-r, r, r)$, $(r, -r, r)$, $(r, r, -r)$ respectively. If AA', BB', CC' are concurrent, we write the common point as convex combinations of A, A'; B, B'; C, C':

$$(1-x)(a,0,0)+x(-r,r,r) = (1-y)(0,b,0)+y(r,-r,r) = (1-z)(0,0,c)+z(r,r,r)$$

for real numbers x, y, z. This is equivalent to

$$(a-(a+r)x, rx, rx) = (ry, b-(b+r)y, ry) = (rz, rz, c-(c+r)z).$$

From the third components of the first two triplets, we have $x = y$. Similarly, $z = x$. Therefore, $x = y = z$. It follows that $a - (a + r)x = rx$, and $x = \frac{a}{a+2r}$. Similarly, $y = \frac{b}{b+2r}$ and $z = \frac{c}{c+2r}$. Now, $x = y = z$ again implies

$$\frac{a}{a+2r} = \frac{b}{b+2r} = \frac{c}{c+2r} \Rightarrow \frac{a}{2r} = \frac{b}{2r} = \frac{c}{2r} \Rightarrow a = b = c.$$

A stronger converse clearly holds: if $a = b = c$, then $D' = \left(\frac{2a-3r}{3}, \frac{2a-3r}{3}, \frac{2a-3r}{3}\right)$ and the four lines AA', BB', CC', DD' concur at $\left(\frac{ar}{a+2r}, \frac{ar}{a+2r}, \frac{ar}{a+2r}\right)$. A simple calculation shows that $r = \frac{a}{3+\sqrt{3}}$.

Except for one incomplete solution, we received no other submissions.

3681. [2011 : 455, 457] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let D, E, and F be the points where the incircle of ΔABC touches the sides. Let Z be the Gergonne point (where AD, BE, and CF concur), and let M be the midpoint of BC. Define T to be the tangency point of the incircle with the circle through B and C that is tangent to it, and let the common tangent line at that point intersect AC at S. Prove that AB, SZ, and ME are concurrent.

Solution by Titu Zvonaru, Comăneşti, Romania.

The statement of the result is not quite correct. We shall prove that the lines AB, SZ, and ME are *concurrent or parallel*. We use standard notation: r denotes the inradius of ΔABC, while $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. Then $BF = BD = s-b$, $CD = CE = s-c$, $AE = AF = s-a$.

Van Aubel's theorem asserts that $\frac{CZ}{ZF} = \frac{CE}{EA} + \frac{CD}{DB}$, hence

$$\frac{CZ}{ZF} = \frac{s-c}{s-a} + \frac{s-c}{s-b} = \frac{c(s-c)}{(s-a)(s-b)}. \tag{1}$$

If $b = c$ then $ST||BC$ and, because AM would be perpendicular to BC, $AM = \frac{2rs}{a}$ and

$$\frac{CS}{SA} = \frac{2r}{AM-2r} = \frac{a}{s-a}. \tag{2}$$

Otherwise, we can assume that $b > c$ and let $P = ST \cap BC$. Using the power of P with respect to the two circles we see that $PD^2 = PT^2 = PB \cdot PC$, or equivalently, $(PC - DC)^2 = PC(PC - BC)$, $(PC - (s-c))^2 = PC(PC - a)$, and finally,

$$PC = \frac{(s-c)^2}{b-c}.$$

Let s' be the semiperimeter of ΔPCS and $x = SE$; observe that $s' = PC + x$. Making use of the areas of ΔPCS and ΔABC we have $rs' = \frac{PC \cdot SC \cdot \sin C}{2}$, which is equivalent to

$$\left(\frac{(s-c)^2}{b-c} + x\right) \cdot r = \frac{(s-c)^2}{b-c}(x+s-c)\frac{\sin C}{2},$$

$$\left(\frac{(s-c)^2}{b-c} + x\right) \cdot abr = \frac{(s-c)^2}{b-c}(x+s-c)\frac{ab\sin C}{2}, \text{ and}$$

$$\left(\frac{(s-c)^2}{b-c} + x\right) \cdot ab = \frac{(s-c)^2}{b-c}(x+s-c)s.$$

Solving for x we obtain

$$SE = x = \frac{ab(s-c)^2 - s(s-c)^3}{s(s-c)^2 - ab(b-c)}.$$

It follows that

$$SC = SE + s - c = \frac{ab(s-c)^2 - ab(b-c)(s-c)}{s(s-c)^2 - ab(b-c)} = \frac{ab(s-c)(s-b)}{s(s-c)^2 - ab(b-c)},$$

and

$$SA = b - CS = \frac{b\left(s(s-c)^2 - ab(b-c) - a(s-c)(s-b)\right)}{s(s-c)^2 - ab(b-c)}.$$

With a few lines of algebra (and recalling that $2s = a + b + c$), one can show that the numerator of SA reduces to $b(s-a)(s-b)^2$. It follows that

$$\frac{CS}{SA} = \frac{a(s-c)}{(s-a)(s-b)}. \tag{3}$$

Finally, observe that if $a = c$ then E would be the midpoint of CA and, thus, ME would be parallel to AB. Using equation (1) with either (2) or (3), we find $a = c$ implies that $\frac{CZ}{ZF} = \frac{a}{s-b} = \frac{CS}{SA}$, whence we would have AB and ME also parallel to SZ. So, let us assume that $a \neq c$ and let Q be the point where AB and ME intersect. Applying Menelaus's theorem to ΔABC, we get $\frac{AQ}{QB} \cdot \frac{BM}{MC} \cdot \frac{CE}{EA} = -1$ if and only if

$$\frac{QA}{QA + c} = \frac{s-a}{s-c}, \text{ and, therefore, } QA = \frac{c(s-a)}{a-c}.$$

Consequently, $QF = \frac{c(s-a)}{a-c} + s - a = \frac{a(s-a)}{a-c}$, and (because Q is necessarily outside the segment AB we require the negative sign for applying the converse of Menelaus's theorem using directed distances)

$$\frac{AQ}{QF} = -\frac{c}{a}. \tag{4}$$

We are now ready to apply the converse of Menelaus's theorem to ΔAFC. When $b \neq c$ we use (4), (1), and (3) to obtain

$$\frac{AQ}{QF} \cdot \frac{FZ}{ZC} \cdot \frac{CS}{SA} = -\frac{c}{a} \cdot \frac{(s-a)(s-b)}{c(s-c)} \cdot \frac{a(s-c)}{(s-a)(s-b)} = -1.$$

We conclude that the points Q, Z, S are collinear, which means that the lines AB, SZ, and ME are concurrent in Q. The conclusion continues to hold when $b = c$ if we replace (3) by (2).

Also solved by MICHEL BATAILLE, Rouen, France and the proposer.

Computer graphics strongly suggest that the result continues to hold when the incircle of ΔABC is replaced by the excircle opposite vertex A, and even in the intermediate case in which $BF || CE$. See the comments following problem 3684.

3682. [2011 : 455, 457] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers such that $a^2+b^2+c^2+d^2 = 1$. Prove that

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-cd} + \frac{1}{1-da} + \frac{1}{1-bd} + \frac{1}{1-ac} \le 8.$$

I. Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Michel Bataille, Rouen, France; and the proposer.

We have the following inequality with its analogues for the other pairs of variables:

$$\begin{aligned}\frac{1}{1-ab} &= 1+\frac{ab}{1-ab} = 1+\frac{ab}{a^2+b^2+c^2+d^2-ab}\\ &= 1+\frac{2ab}{2(c^2+d^2)+a^2+b^2+(a-b)^2}\\ &\le 1+\frac{2ab}{(c^2+d^2+a^2)+(c^2+d^2+b^2)}\\ &\le 1+\frac{1}{2}\left[\frac{(a+b)^2}{(c^2+d^2+a^2)+(c^2+d^2+b^2)}\right]\\ &\le 1+\frac{1}{2}\left[\frac{a^2}{c^2+d^2+a^2}+\frac{b^2}{c^2+d^2+b^2}\right].\end{aligned}$$

The second inequality is a consequence of the Arithmetic Geometric Means Inequality and the third results from the identity

$$\frac{a^2}{u}+\frac{b^2}{v}-\frac{(a+b)^2}{u+v} = \frac{(av-bu)^2}{uv(u+v)}.$$

Adding the six inequalities yields the desired result. Equality occurs if and only if $a=b=c=d=\frac{1}{2}$.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

Observe first that $ab \le \frac{1}{2}(a^2+b^2) \le \frac{1}{2}$, with similar inequalities for the other pairs of variables. When $0 \le x \le \frac{1}{2}$, we have that $2(5+16x^2)(1-x)-9 = (1-2x)(1-4x)^2 \ge 0$ so that

$$\frac{1}{1-x} \le \frac{2}{9}(5+16x^2).$$

Replacing x by the products of pairs of the variables and setting $s = a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2$, we find that the left side of the inequality does not exceed

$$\frac{20}{3}+\frac{32}{9}s.$$

Since

$$\begin{aligned}3-8s &= 3(a^2+b^2+c^2+d^2)^2-8s\\ &= (a^2-b^2)^2+(a^2-c^2)^2+(a^2-d^2)^2+(b^2-c^2)^2+(b^2-d^2)^2+(c^2-d^2)^2,\end{aligned}$$

$s \le 3/8$ and the result follows. Equality occurs when each variable equals $\frac{1}{2}$.

One incorrect solution was received.

3683. [2011 : 455, 457] *Proposed by Michel Bataille, Rouen, France.*

Let n be an integer with $n \geq 2$ and z a complex number with $|z| \leq 1$. Prove that

$$\sum_{k=1}^{n} kz^{n-k} \neq 0\,.$$

I. Solution by Dimitrios Koukakis, Kato Apostoloi, Greece.

The sum does not vanish when $z = 1$. When $z \neq 1$, the sum vanishes if and only if

$$0 = z^n + z^{n-1} + \cdots + z - n = (z-1)(z^{n-1} + 2z^{n-2} + \cdots + (n-1)z + n).$$

When $|z| < 1$, then

$$|z^n + z^{n-1} + \cdots + z| \leq |z|^n + |z|^{n-1} + \cdots + |z| < n$$

so the sum does not vanish.

Suppose that $|z| = 1$ and that

$$z^n + z^{n-1} + \cdots + z = n.$$

Taking the complex conjugate and using the fact that $\bar{z} = 1/z$, we obtain the equations

$$1 + z + z^2 + \cdots + z^{n-1} = nz^n$$

and

$$z + z^2 + z^3 + \cdots + z^n = nz^{n+1}.$$

Subtracting each of these equations from the first leads to $z^n - 1 = n(1 - z^n)$ and $n(1 - z^{n+1}) = 0$. Hence $z^n = z^{n+1} = 1$, so that $z = 1$. The result follows.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

For $z \neq 1$, it is equivalent to show that $z^n + z^{n-1} + \cdots + z \neq n$ when $|z| \leq 1$. It is straightforward to establish the result when z is real and satisfies $-1 \leq z < 1$.

Suppose that $|z| \leq 1$ and z is nonreal. Then the arguments of z and z^2 are distinct, so that $|z^2 + z| < |z^2| + |z| \leq 2$. Hence

$$|z^n + z^{n-1} + \cdots + z^3 + z^2 + z| \leq |z|^n + |z|^{n-1} + \cdots + |z|^3 + |z^2 + z| < n$$

and the desired result follows.

III. Solution by Kee-Wai Lau, Hong Kong, China.

The case $|z| < 1$ can be handled as in the first solution. Observe that

$$z^{n+1} - (n+1)z + n = (z-1)^2(z^{n-1} + 2z^{n-2} + \cdots + (n-1)z + n).$$

Suppose that $|z| = 1$ and that

$$z^{n+1} = (n+1)z - n.$$

With $z = \cos\theta + i\sin\theta$, this leads to

$$\cos(n+1)\theta = (n+1)\cos\theta - n$$
$$\sin(n+1)\theta = (n+1)\sin\theta.$$

Squaring and adding these equations yields that

$$1 = (n+1)^2 - 2n(n+1)\cos\theta + n^2 = 2n(n+1)(1-\cos\theta) + 1 \geq 1.$$

Hence $\theta = 0$, so that $z = 1$. Since the sum of the problem does not vanish when $z = 1$, we see that it cannot vanish whenever $|z| \leq 1$.

IV. Solution by the proposer.

The case $|z| < 1$ can be handled as in the first solution. Suppose that $z = e^{i\theta}$ where $0 < \theta < 2\pi$ and that $z^{n+1} = (n+1)z - n$. Then

$$1 = |z|^{n+1} = |(n+1)z - n| = (n+1)\left|e^{i\theta} - \frac{n}{n+1}\right|.$$

But

$$\left|e^{i\theta} - \frac{n}{n+1}\right|^2 - \frac{1}{(n+1)^2} = \left(\cos\theta - \frac{n}{n+1}\right)^2 + \sin^2\theta - \frac{1}{(n+1)^2}$$
$$= \frac{2n}{n+1}(1-\cos\theta) > 0.$$

Hence $|e^{i\theta} - n/(n+1)| > 1/(n+1)$ and we get a contradiction. The result follows.

V. Solution by John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA.

The sum does not vanish when $z = 0$. Otherwise, set $z = 1/w$. It is equivalent to show that all the zeros of $g(w) = 1 + 2w + 3w^2 + \cdots + (n-1)w^{n-2} + nw^{n-1}$ lie inside the open unit disc $|w| < 1$. Now $g(z)$ is the derivative of the function $f(z) = 1 + z + z^2 + \cdots + z^n$ whose zeros are the $(n+1)$th roots of unity distinct from 1, and so are simple and lie on the unit circle $|w| = 1$.

By the Gauss-Lucas Theorem [1, 2], the zeros of $g(w)$ are contained within the closed polygon whose vertices are the zeros of $f(w)$. Since none of the zeros of $f(w)$ are zeros of $g(w)$, all the zeros of $f(w)$ are contained within the open unit disc.

References

[1] Peter Borwein and Tamás Erdelyi, *Polynomials and Polynomial Inequalities.* Springer, New York, 1995. page 18

[2] Victor V. Prasolov, *Polynomials.* Springer, Berlin, Heidelberg, 2004. page 13

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA. Three incomplete solutions were received.

3684. [2011 : 455, 457] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Given two circles that are internally tangent at T, let the chord BC of the outer circle be tangent to the inner circle at D. Let the second tangents from B and C touch the inner circle at F and E respectively, and define $J = EF \cap DT$ and $Z = BE \cap CF$. Prove that

(a) JZ intersects BC at its midpoint, and

(b) TD bisects $\angle BTC$.

Comment. This result allows for a solution to a special case of the Problem of Apollonius: Construct a circle through two given points that is tangent to a given circle which, itself, is tangent to the line joining the given points.

Solution by Titu Zvonaru, Cománeşti, Romania.

If the common tangent at T is parallel to the chord BC, then the quadrilateral $EFBC$ is an isosceles trapezoid, and both parts of the problem are easily verified. We therefore assume that the tangent at T meets BC at P and EC at S. The argument also becomes easier should DT be parallel to EC (in which case E is the midpoint of SC). We therefore define

$$V = EF \cap BC \text{ and } Y = DT \cap EC.$$

Part (a). Let us now consider the case in which CE meets BF at a point A and, moreover, the inner circle is the incircle of ΔABC. Because the outer circle is the circle through B and C that is tangent to the incircle, we are working with the configuration of problem **3681**. That means we can use the results obtained there, specifically,

$$\begin{aligned} PC &= \frac{(s-c)^2}{b-c} \text{ (and, therefore, } PD = PC - (s-c) = \frac{(s-b)(s-c)}{b-c}); \\ SE &= \frac{ab(s-c)^2 - s(s-c)^3}{s(s-c)^2 - ab(b-c)} = \frac{(s-a)(s-b)(s-c)^2}{s(s-c)^2 - ab(b-c)}; \\ SC &= \frac{ab(s-b)(s-c)}{s(s-c)^2 - ab(b-c)}; \end{aligned}$$

hence,

$$\frac{ES}{SC} = \frac{(s-a)(s-c)}{ab}.$$

By van Aubel's sum theorem,

$$\frac{BZ}{ZE} = \frac{BF}{FA} + \frac{BD}{DC} = \frac{s-b}{s-a} + \frac{s-b}{s-c} = \frac{b(s-b)}{(s-a)(s-c)}. \qquad (1)$$

We now apply Menelaus's theorem three times.
(i) To the transversal VEF of ΔBCA:

$$-1 = \frac{BV}{VC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{BV}{VC} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b};$$

thus, $\frac{VB}{VC} = \frac{s-b}{s-c}$. Because $VB = VM - \frac{a}{2}$ and $VC = VM + \frac{a}{2}$, we calculate that $VM = \frac{a^2}{2(b-c)}$, whence

$$\frac{BM}{MV} = \frac{c-b}{a}. \tag{2}$$

Furthermore, we have

$$VD = VB + (s-b) = \frac{a^2}{2(b-c)} - \frac{a}{2} + s - b = \frac{2(s-b)(s-c)}{b-c}.$$

(ii) To the transversal YDT of ΔSCP:

$$\frac{SY}{YC} \cdot \frac{CD}{DP} \cdot \frac{PT}{TS} = -1.$$

Because $DP = PT$, we deduce that $YS = \frac{YC \cdot TS}{DC} = \frac{(YS+SC)\cdot SE}{EC}$, whence

$$YS = \frac{SC \cdot SE}{EC - SE}.$$

It follows that

$$YE = \frac{SC \cdot SE}{EC - SE} + SE = \frac{SE \cdot (SC - SE + EC)}{EC - SE} = \frac{2SE \cdot EC}{EC - SE},$$

and

$$YC = YE + EC = \frac{EC \cdot (2SE + EC - SE)}{EC - SE} = \frac{EC \cdot SC}{EC - SE}.$$

Consequently,

$$\frac{EY}{YC} = \frac{2ES}{SC} = -\frac{2(s-a)(s-c)}{ab}.$$

And (iii) to the transversal YDJ of ΔECV:

$$-1 = \frac{EY}{YC} \cdot \frac{CD}{DV} \cdot \frac{VJ}{JE} = -\frac{2(s-a)(s-c)}{ab} \cdot \frac{s-c}{\frac{2(s-b)(s-c)}{b-c}} \cdot \frac{VJ}{JE};$$

thus,

$$\frac{VJ}{JE} = \frac{ab(s-b)}{(s-a)(s-c)(b-c)}. \tag{3}$$

Using equations (3), (1), and (2) we deduce that

$$\frac{VJ}{JE} \cdot \frac{EZ}{ZB} \cdot \frac{BM}{MV} = \frac{ab(s-b)}{(s-a)(s-c)(b-c)} \cdot \frac{(s-a)(s-c)}{b(s-b)} \cdot \frac{c-b}{a} = -1.$$

The converse of Menelaus's theorem applied to ΔVEB and the points J, Z and M implies that J, Z, and M are collinear, which completes part (a).
Part (b). Because $\angle TPB = \angle TPC$ and $\angle PTB = \angle TCP$, the triangles PBT and PTC are similar, so that

$$\frac{TB}{CT} = \frac{PT}{PC} = \frac{PD}{PC} = \frac{(s-b)(s-c)}{b-c} \cdot \frac{b-c}{(s-c)^2} = \frac{s-b}{s-c} = \frac{BD}{DC}.$$

In words, D is the point that divides the side BC of ΔTBC internally in the ratio equal to that of the other two sides, so that it lies on the bisector of the angle between those two sides; that is, TD bisects $\angle BTC$.

Also solved by MICHEL BATAILLE, Rouen, France and the proposer.

Disclaimer: The statement of the problem allows for the inner circle to be the excircle that is tangent to side BC of the triangle ABC formed by the lines BC, CE, BF. Indeed, it even allows for the lines BF and CE to be parallel so that A would be at infinity. None of the submitted solutions addressed these possibilities. Although it would probably be easy in each case to modify the featured solution, the details seem to involve more than simply using directed distances and angles. It seems prudent, therefore, for us to follow Euclid's approach and present only a typical case of the problem, leaving the remaining cases to the reader. Of course, Euclid gave the special cases to his grad students to check the details. Unfortunately, this editor has no grad students, so I had to rely on the graphics program Cinderella *to confirm the claims. The computer turned up no surprises, so Problem* **3684** *is probably correct as stated; it has been proved, however, only for the arrangement described by the earlier Problem* **3681**.

3685. [2011 : 455, 458] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0,1] \to (0,\infty)$ be a bounded function which is continuous at 0. Find the value of

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{f\left(\frac{1}{1}\right)}+\sqrt[n]{f\left(\frac{1}{2}\right)}+\cdots+\sqrt[n]{f\left(\frac{1}{n}\right)}}{n}\right)^n .$$

Solution by Michel Bataille, Rouen, France.

The answer is $f(0)$. We require two results, the power mean inequality for $0 < r < 1$, $a_k > 0$,

$$\left(\prod_{k=1}^{n} a_k\right)^{1/n} \le \left(\frac{a_1^r + a_2^r + \cdots + a_n^r}{n}\right)^{1/r} \le \frac{a_1 + a_2 + \cdots + a_n}{n},$$

and the Cesaro limit result for a convergent sequence $\{b_n\}$,

$$\lim_{n\to\infty}\frac{b_1 + b_2 + \cdots + b_n}{n} = \lim_{n\to\infty} b_n .$$

Let

$$U_n = \left(\frac{\sqrt[n]{f(1/1)}+\sqrt[n]{f(1/2)}+\cdots+\sqrt[n]{f(1/n)}}{n}\right)^n,$$

$$A_n = \frac{f(1/1) + f(1/2) + \cdots + f(1/n)}{n},$$

and

$$G_n = (f(1/1)\cdot f(1/2)\cdot \cdots \cdot f(1/n))^{1/n}.$$

Using the power mean inequality with $r = 1/n$, we obtain $G_n \le U_n \le A_n$. Since $\lim_{n\to\infty} f(1/n) = f(0) > 0$, we can apply the Cesaro result with $b_n = f(1/n)$ to obtain that $\lim_{n\to\infty} A_n = f(0)$ and $\lim_{n\to\infty} \ln G_n = \ln f(0)$. By the squeeze principle $\lim_{n\to\infty} U_n = f(0)$.

Also solved by DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer. Schlosberg used the harmonic mean instead of the geometric mean.

3686. [2011 : 456, 458] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be real numbers such that $abc = 1$. Show that

$$\left(a - \frac{1}{a} + b - \frac{1}{b} + c - \frac{1}{c}\right)^2 \le 2\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right).$$

I. Solution by Arkady Alt, San Jose, CA, USA; and Titu Zvonaru, Cománeşti, Romania(independently).

Let $x = a+b+c$ and $y = ab+bc+ca$. Since $abc = 1$, the difference between the two sides of the inequality is

$$\begin{aligned}
&2(a^2+1)(b^2+1)(c^2+1) - (a+b+c-bc-ca-ab)^2 \\
&\quad = 2(2+a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2) - (a+b+c-ab-bc-ca)^2 \\
&\quad = 2(2+y^2-2x+x^2-2y) - (x-y)^2 = (x^2+y^2+2xy-4x-4y+4) \\
&\quad = (x+y-2)^2 \ge 0.
\end{aligned}$$

Equality occurs if and only if $a+b+c+ab+bc+ca-2 = 0$.

II. Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl, NRW, Germany; Kee-Wai Lau, Hong Kong, China; Salem Malikić, student, Simon Fraser University, Burnaby, BC; and Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA (independently).

Let

$$f(a,b) = 2(a^2+1)(b^2+1)(a^2b^2+1) - (a^2b+ab^2+1-a-b-a^2b^2)^2.$$

Replacing c by $1/ab$, we find that the difference of the two sides of the inequality is

$$\begin{aligned}
a^{-2}b^{-2}f(a,b) &= a^{-2}b^{-2}(1+a+b-2ab+a^2b+ab^2+a^2b^2)^2 \\
&= (c+ac+bc-2+a+b+ab)^2 \ge 0,
\end{aligned}$$

from which the result follows, with the foregoing condition for equality.

III. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy(independently).

We can determine nonzero real values u, v, w for which $a = v/w$, $b = w/u$ and $c = u/v$. Then

$$\begin{aligned} a - \frac{1}{a} + b - \frac{1}{b} + c - \frac{1}{c} &= (uvw)^{-1}(uv^2 - uw^2 + vw^2 - vu^2 + wu^2 - wv^2) \\ &= (uvw)^{-1}(u-v)(v-w)(w-u) \end{aligned}$$

and

$$\begin{aligned} &2\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right) \\ =&(uvw)^{-2}[(1^2+1^2)(v^2+w^2)][(u^2+v^2)(u^2+w^2)] \\ =&(uvw)^{-2}[(v+w)^2+(v-w)^2][(u^2+vw)^2+u^2(v-w)^2] \\ =&(uvw)^{-2}[((v+w)(u^2+vw)+u(v-w)^2)^2+(u(v+w)(v-w)-(v-w)(u^2+vw))^2] \\ =&(uvw)^{-2}[(uv(u+v)+vw(v+w)+wu(u+v)-2uvw)^2+(v-w)^2(u-v)^2(u-w)^2]. \end{aligned}$$

The desired inequality follows; equality occurs if and only if

$$uv(u+v) + vw(v+w) + wu(u+w) - 2uvw = 0,$$

which is equivalent to $ac + a + ba + b + bc + c - 2 = 0$.

Also solved by DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece; and the proposer. Several solvers pointed out that equality occurs for infinitely many triples. Since the condition can be written as $(a + a^{-1}) + (b + b^{-1}) + (c + c^{-1}) = 2$, it is clear that not all of a, b, c can be positive. McCartney found this in a more precise way by noting that

$$f(a,b) - 16 = (1+a)(1+b)(1+ab)(1+a+b-6ab+a^2b+ab^2+a^2b^2)$$

and observing that the last factor is positive for positive a, b, c since $6ab \le 1+a+b+a^2b+ab^2+a^2b^2$ by the Arithmetic-Geometric Means Inequality. Since $abc = 1$, the condition for equality can be written as

$$a(a+1)b^2 + (a-1)^2b + (a+1) = 0.$$

This is a quadratic equation in b with discriminant $a^4-8a^3-2a^2-8a+1 = (a^2-1)^2-8a(a^2+1)$. Select any negative value of a to get a positive discriminant, solve the quadratic for b and set $c = a^{-1}b^{-1}$ to obtain equality.

3687. [2011 : 456, 458] *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{\infty} \frac{k^n}{k!}\left(k + 1 - \frac{1}{k!}\int_1^{\infty} e^{-t}t^{k+1}dt\right) = \sum_{k=0}^{n} \frac{S(n,k)}{k+2},$$

where k^n is taken to be 1 for $k = n = 0$ and $S(n,k)$ are the Stirling numbers of the second kind that are defined by the recursion

$$S(n,m) = S(n-1,m-1) + mS(n-1,m), S(n,0) = \delta_{0,n}, S(n,n) = 1\,.$$

[*Ed.: The proposer's original problem erroneously had an extra term $k!$ in the denominator that was not caught by the editorial board. As a result, no other* *solutions were received. The corrected version of the problem will appear in a future problem set.]*

3688. [2011 : 540, 542] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let $T_n(x)$ be the Chebyshev polynomial of the first kind defined by the recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 1$ and the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. Find all positive integers n such that

$$T_n(x) \geq (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}, \quad x \in [1,\infty).$$

[*Ed.: Note problem* **3585** *was originally printed with the wrong inequality.*]

Solution by Michel Bataille, Rouen, France.

For $n \geq 1$, let $P_n(x) = (2^{n-2}+1)x^n - 2^{n-2}x^{n-1}$. We will show that $T_n(x) \geq P_n(x)$ holds for all $x \in [1,\infty)$ if and only if $n \in \{2,3,4,5,6,7\}$.

First, we notice that since $T_1(x) = x$ and $P_1(x) = \frac{3}{2}x - \frac{1}{2}$ then $T_1(x) < P_1(x)$ for $x \in (1,\infty)$, so we may assume $n \geq 2$ in what follows. It is well-known that $T_n(\cos\theta) = \cos(n\theta)$ for $\theta \in \mathbb{R}$. Using the fact that $\cos(n\theta)$ is the real part of $(\cos\theta + i\sin\theta)^n$ and the binomial theorem, it is readily obtained that

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k.$$

It follows that

$$T_n(x) - P_n(x) = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k - 2^{n-2}x^{n-1}(x-1) = x^{n-1}(x-1)\delta_n(x),$$

where

$$\delta_n(x) = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} \left(1+\frac{1}{x}\right)^k \left(1-\frac{1}{x}\right)^{k-1} - 2^{n-2}.$$

Therefore, $T_n(x) - P_n(x)$ has the same sign as $\delta_n(x)$ for $x > 1$. By induction, it is easy to show that $\delta_n(1) < 0$ for all $n \geq 8$ and by continuity, $\delta_n(1) < 0$ for $x > 1$ sufficiently close to 1. Thus, $T_n(x) \geq P_n(x)$ for all $x \in [1,\infty)$ can hold only if $n \in \{2,3,4,5,6,7\}$.

Now, since

$$x\delta_2(x) = 1, \qquad x\delta_3(x) = x+3,$$
$$x^3\delta_4(x) = x(3x^2-1) + (7x^2-1), \qquad x^3\delta_5(x) = x(7x^2-5) + (15x^2-5),$$

and

$$\begin{aligned} x^5\delta_6(x) &= 15x^5 + 31x^4 - 17x^3 - 17x^2 + x + 1 \\ &\geq 46x^4 - 17x^3 - 17x^2 + x + 1 = 17x^3(x-1) + x^2(29x^2-17) + x + 1, \end{aligned}$$

then $\delta_n(x) > 0$ for all $x > 1$ and $n = 2, 3, 4, 5, 6$. Finally, if $n = 7$, we obtain $x^5\delta_7(x) = \phi(x)$ where $\phi(x) = 31x^5 + 63x^4 - 49x^3 - 49x^2 + 7x + 7$. Since $\phi^{'}(x) = 155x^4 + 252x^3 - 147x^2 - 98x + 7 > 0$ for $x \geq 1$, we have $\phi(x) > \phi(1) > 0$ for $x > 1$ and $\delta_7(x) > 0$ for all $x > 1$ again. This completes the proof.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

3689. [2011 : 540, 543] *Proposed by Ivaylo Kortezov, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria.*

In a group of n people, each one has a different book. We say that a pair of people performs a *swap* if they exchange the books they currently have. Find the least possible number $E(n)$ of swaps such that each pair of people has performed at least one swap and at the end each person has the book he or she had at the start.

Solution by M. A. Prasad, India; expanded slightly by the editor.

We show that

$$E(n) = \begin{cases} \frac{n(n-1)}{2}, & \text{if } n \equiv 0, 1 \pmod 4 \\ \frac{n(n-1)}{2} + 1, & \text{if } n \equiv 2, 3 \pmod 4 . \end{cases}$$

To avoid triviality, we assume that $n \geq 2$. Since there are $\binom{n}{2}$ pairs of people $E(n) \geq \binom{n}{2} = \frac{n(n-1)}{2}$. Furthermore, since everyone gets his/her book back at the end, E(n) must be even.

[*Ed.: Using the terminology and well known facts from the theory of permutation groups, E(n) is the minimum number of transpositions (2-cycles) performed on the set $\{1, 2, 3, \ldots, n\}$ such that the product of which yields the identity permutation σ, and every transposition (i, j) must appear at least once in the product for all $i, j = 1, 2, \ldots, n$ with $i \neq j$. It is well known that a transposition is odd and σ is even. Hence there must be an even number of transpositions in the product.*]

Therefore, if $\frac{n(n-1)}{2}$ is odd, then $E(n) \geq \frac{n(n-1)}{2} + 1$.

Clearly $E(2) = 2$ and it is easy to see that $E(3) = 4$. We label the n people by $1, 2, \ldots, n$ and for $i, j = 1, 2, \ldots, n$ with $i \neq j$ we use (i, j) to denote the swap between i and j. For $n = 4$, the sequence of swaps $(1, 2)$, $(1, 3)$, $(2, 4)$, $(1, 4)$, $(2, 3)$, $(3, 4)$ (performed from left to right) shows that $E(4) = 6$ and for $n = 5$, the sequence $(1, 5)$, $(1, 2)$, $(2, 5)$, $(3, 5)$, $(3, 4)$, $(4, 5)$, $(2, 3)$, $(1, 4)$, $(1, 3)$, $(2, 4)$ shows that $E(5) = 10$. We now proceed by induction to prove our claim.

Suppose that $E(n) = \frac{n(n-1)}{2} + \mathcal{E}$ for some $n \geq 5$ where $\mathcal{E} = 0$ or 1 depending on whether $\frac{n(n-1)}{2}$ is even or odd. We show that

$$\begin{aligned} E(n+4) &= \frac{n(n-1)}{2} + \mathcal{E} + 4n + 6 \\ &= \frac{(n+4)(n+3)}{2} + \mathcal{E}. \end{aligned}$$

We denote the $n + 4$ books by $b_1, b_2, \ldots, b_{n+4}$ and break up the swaps into six steps as follows:

(i) Performing the $\frac{n(n-1)}{2} + \mathcal{E}$ swaps between $5, 6, \ldots, n+4$ to ensure that at the end each of these people has his/her own book.

(ii) Performing the swaps $(1,5)$, $(1,6)$, $\ldots$, $(1, n+4)$ so the books are permuted to $(b_{n+4}, b_2, b_3, b_4, b_1, b_5, b_6, \ldots, b_{n+3})$.

(iii) Performing the swap $(1,2)$ to get $(b_2, b_{n+4}, b_3, b_4, b_1, b_5, b_6, \ldots, b_{n+3})$.

(iv) Performing the swaps $(2, n{+}4)$, $(2, n{+}3)$, $\ldots$, $(2,5)$ so the books are permuted to $(b_2, b_1, b_3, b_4, b_5, \ldots, b_{n+4})$.

(v) Repeat the operations in (ii), (iii) and (iv) with 1 replaced by 3 and 2 replaced by 4 (that is, we perform $(3,5)$, $(3,6)$, $\ldots$, $(3, n+4)$ followed by $(3,4)$ and then $(4, n{+}4)$, $(4, n{+}3)$, $\ldots$, $(4,5)$.) The books are permuted as $(b_2, b_1, b_4, b_3, b_5, \ldots, b_{n+4})$.

(vi) Performing the swaps $(2,3)$, $(1,4)$, $(1,3)$ and $(2,4)$ would then bring all the books back to their original order.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and the proposer. There was an incorrect solution which was caused by misunderstanding the statement of the problem.

3690. [2011 : 540, 543; 2012 : 23, 25] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be three distinct positive real numbers with $a+b+c=1$. Show that

$$(5x^2 - 6xy + 5y^2)(a^3 + b^3 + c^3) + 12(x^2 - 3xy + y^2)abc > (x-y)^2$$

for all real numbers x and y, not both zero.

Solution by Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA.

The inequality to be proven is equivalent to

$$f(x,y) \equiv \left(5 \sum_{\text{cyclic}} a^3 + 12abc - 1\right)(x-y)^2 + \left(\sum_{\text{cyclic}} a^3 - 3abc\right) 4xy > 0.$$

Let C denote the coefficient of $(x-y)^2$. Then

$$C = 5\sum_{\text{cyclic}} a^3 + 12abc - 1 = \left(\sum_{\text{cyclic}} a^3 - 3abc\right) + \left(4\sum_{\text{cyclic}} a^3 + 15abc - 1\right).$$

By the AM-GM inequality, we have $\displaystyle\sum_{\text{cyclic}} a^3 - 3abc > 0$ since a, b, and c are positive and distinct.

Furthermore, since

$$1 = \left(\sum_{\text{cyclic}} a\right)^3 = \sum_{\text{cyclic}} a^3 + 3\sum_{\text{cyclic}} (a^2b + ab^2) + 6abc,$$

we have

$$4\sum_{\text{cyclic}} a^3 + 15abc - 1 = 3\left(\sum_{\text{cyclic}} a^3 - \sum_{\text{cyclic}} (a^2b + ab^2) + 3abc\right) > 0$$

by Schur's Inequality. Hence

$$\begin{aligned} f(x,y) = &\left[\left(\sum_{\text{cyclic}} a^3 - 3abc\right) + 3\left(\sum_{\text{cyclic}} a^3 - \sum_{\text{cyclic}} (a^2b+ab^2) + 3abc\right)\right](x-y)^2 \\ &\qquad + \left(\sum_{\text{cyclic}} a^3 - 3abc\right)4xy \\ = &\ 3\left(\sum_{\text{cyclic}} a^3 - \sum_{\text{cyclic}} (a^2b+ab^2) + 3abc\right)(x-y)^2 \\ &\qquad + \left(\sum_{\text{cyclic}} a^3 - 3abc\right)(x+y)^2 > 0 \end{aligned}$$

since if $x - y = x + y = 0$, then $x = y = 0$.

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; RADOUAN BOUKHARFANE, Polytechnique de Montréal, Montréal, PQ; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; DANIEL VACARU, Pitesti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer.

Almost all of the submitted solutions used Schur's Inequality and many of which are similar to the featured solution.