SKOLIAD No. 142

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Skoliad has joined **Mathematical Mayhem** *which is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time Skoliad will be discontinued in* **Crux**. *New Skoliad problems, and their solutions, will appear in* **Mathematical Mayhem** *when it is relaunched.*

In this issue we present the solutions to the Swedish Junior High School Mathematics Contest, Final Round, 2010/2011, given in Skoliad 136 at [2011 : 409–410].

1. The year 2010 is divisible by three consecutive primes. Find the last year before that with this property.

Solution by Lucy Yuan, student, New Westminster Secondary School, New Westminster, BC.

The first few primes are 2, 3, 5, 7, 11, 13, and 17. The desired year cannot be divisible by the last three of these, since $11 \times 13 \times 17 = 2431$, which is too large. Likewise, any triple of larger primes can be discarded.

If the desired year is divisible by 2, 3, and 5, then it is divisible by $2 \times 3 \times 5 = 30$. Now $2010 \div 30 = 67$, so the latest such year prior to 2010 is $66 \times 30 = 1980$.

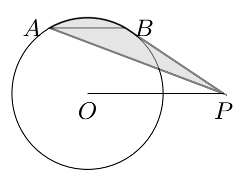
If the desired year is divisible by 3, 5, and 7, then it is divisible by $3 \times 5 \times 7 = 105$. Since $2010 \div 105 \approx 19.1$, the latest such year prior to 2010 is $19 \times 105 = 1995$ (which is closer than 1980).

Similarly, $5 \times 7 \times 11 = 385$ and $2010 \div 385 \approx 5.2$, so the latest year divisible by those three primes is $5 \times 385 = 1925$.

Finally, $7 \times 11 \times 13 = 1001$ and $2010 \div 1001 \approx 2.01$, so the latest year divisible by those three primes is $2 \times 1001 = 2002$.

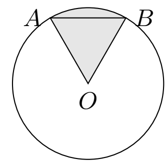
Of the four years found, 1980, 1995, 1925, and 2002, the one closest to 2010 is 2002.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; MAGGIE LIU, student, Burnaby Central Secondary School, Burnaby, BC; SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.



2. Draw a line from the centre, O, of a circle with radius r to a point, P, outside the circle. Then choose two points, A and B, on the circle such that AB has length r and is parallel with OP. Find the area of the shaded region.

Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.



Using AB as the base for both triangles, it is easy to see that $\triangle ABP$ and $\triangle ABO$ have the same area. Therefore the shaded region in the problem has the same area as the shaded region in the figure on the right.

Since $|AB| = r$, the radius of the circle, $\triangle ABO$ is equilateral, so $\angle AOB = 60^\circ$, and the shaded sector is $\frac{60}{360} = \frac{1}{6}$ of the circle. Therefore the area of the shaded sector is $\frac{1}{6}\pi r^2$.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.

3. Five distinct positive numbers are given. No matter which two of them you choose, one divides the other. The sum of the five numbers is a prime. Show that one of the five numbers is 1.

Solution by Andrew Tao, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA.

Let x denote the smallest of the five positive integers. Since no number can divide a smaller number, x divides each of the other four numbers. Therefore x divides the sum of the five numbers. Since the sum is a prime, either x is the sum or $x = 1$. Since the numbers are positive, x cannot equal the sum, so $x = 1$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.

4. A large cube consists of eight identical smaller cubes. The faces of each of the smaller cubes bear the numbers 3, 3, 4, 4, 5, and 5 such that opposite faces bear the same number. Assign to each face of the large cube the sum of the four visible numbers. Show that the numbers assigned to the faces of the large cube cannot be six consecutive integers.

Solution by Anna Verkhovskaya, student, John Ware Junior High, Calgary, AB.

Each of the smaller cubes sits at a corner of the large cube. Therefore three faces of each smaller cube are visible, and none of these faces are opposite each other. Thus, each of the smaller cubes shows the numbers 3, 4, and 5, so each of the smaller cubes contribute $3+4+5 = 12$ to the sum of the numbers on the faces of the large cube. Therefore the sum of the numbers on the faces of the large cube is $8 \times 12 = 96$.

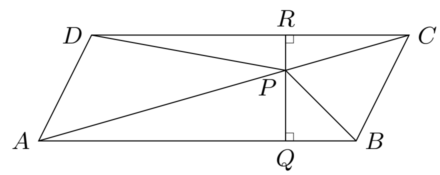
Since any six consecutive integers contain three odd and three even integers, their sum must be an odd number. As 96 is not odd, it cannot be the sum of six consecutive integers, and therefore the sides of the cubes cannot show six consecutive integers.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA.

5. The parallelogram $ABCD$ has area 12. The point P is on the diagonal AC. The area of $\triangle ABP$ is one third of the area of $ABCD$. Find the area of $\triangle CDP$.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Let Q and R be the feet of the perpendiculars to P from AB and CD, respectively. The area of $ABCD$ is $|AB||QR| = 12$, and the area of $\triangle ABP$ is $\frac{1}{2}|AB||QP| = \frac{1}{3}\cdot 12 = 4$, so $|AB||QP| = 8$. Therefore,

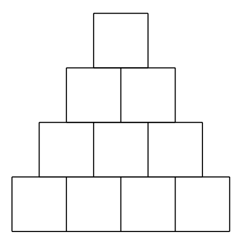


$$\frac{|QP|}{|QR|} = \frac{|AB||QP|}{|AB||QR|} = \frac{8}{12} = \frac{2}{3},$$

so $|QP| = \frac{2}{3}|QR|$. Since $|QP| + |PR| = |QR|$, it follows that $|PR| = \frac{1}{3}|QR| = \frac{1}{2}|QP|$. Thus, the area of $\triangle CDP$ is $\frac{1}{2}|CD||PR| = \frac{1}{2}|AB|\frac{1}{2}|QP| = \frac{1}{2}(\frac{1}{2}|AB||QP|)$, which is half the area of $\triangle ABP$, so half of 4, so 2.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.

6. Place ten numbers in the grid subject to the following rules:

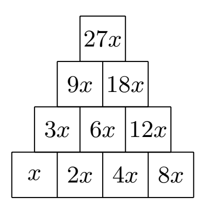


1. For neighbours in the bottom row, the number on the right must be twice as large as the number on the left.
2. Other than in the bottom row, each number is the sum of the two numbers immediately below it.

Find the smallest positive integer that you can place in the bottom left position such that the sum of all ten numbers is a square.

Solution by Alison Tam, student, Burnaby South Secondary School, Burnaby, BC.

Say the number in the bottom left slot is x. Then the numbers in the bottom row must be x, $2x$, $4x$, and $8x$, and you can fill in the entire diagram as shown. The sum of all ten numbers in the diagram is then $90x$.



If $90x = 3^2 \cdot 2 \cdot 5 \cdot x$ is to be a square (and x a positive integer), then $x = 2 \cdot 5 \cdot n^2$, where n is an integer other than zero. The smallest possible value for x occurs, then, when $n = 1$, so $x = 10$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; ANDREW TAO, student, Thomas Jefferson High School for Science at Technology, Alexandria, VA, USA; and ANNA VERKHOVSKAYA, student, John Ware Junior High, Calgary, AB.

This issue's prize for the best solutions goes to Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.