The Barycentric Equation of a Line

Introduction

Barycentric coordinates relative to a triangle $ABC$ constitute a common and convenient tool in plane geometry. A point $P$ has coordinates $(x,y,z)$ (with $x+y+z \neq 0$) if $P$ is the barycentre of $A,B,C$ with respective masses $x,y,z$, that is, if $(x+y+z)\overrightarrow{MP} = x\overrightarrow{MA} + y\overrightarrow{MB} + z\overrightarrow{MC}$ for any point $M$. These coordinates are also called areal coordinates because $x,y,z$ are proportional to the signed areas $[PBC],[PCA],[PAB]$, a nice geometric interpretation of $x,y,z$. In this context, the equation of a line is of the form $ux + vy + wz = 0$ for some real numbers $u,v,w$, not all zero, and this leads to systematic ways of solving problems of collinearity or concurrency. Stepping back, we would like to give a geometric look to the coefficients $u,v,w$ and offer some applications.

Two simple results about $u,v,w$

In this paragraph, we assume that $u,v,w$ are not zero, leaving these special cases to the reader. Let $\ell$ be the line with equation $ux + vy + wz = 0$ and let $\ell$ intersect the sidelines $BC,CA,AB$ at $D,E,F$ respectively. Then we have

$$\frac{v}{w} = -\frac{BD}{DC}, \quad \frac{w}{u} = -\frac{CE}{EA}, \quad \frac{u}{v} = -\frac{AF}{FB}$$  \hspace{1cm} (1)

and, if $A',B',C'$ are the orthogonal projections of $A,B,C$ onto $\ell$,

$$\frac{AA'}{u} = \frac{BB'}{v} = \frac{CC'}{w}$$  \hspace{1cm} (2)

where all distances are signed.

For example, if $(0,\beta,\gamma)$ are the coordinates of $D$, then $v\beta + w\gamma = 0$ and $\beta \overrightarrow{DB} + \gamma \overrightarrow{DC} = \overrightarrow{0}$, hence $w \overrightarrow{DB} = v \overrightarrow{DC}$. The first equality in (1) follows (alternatively, one can observe that $v[DCA] + w[DAB] = 0$ and $\frac{[DAB]}{[DCA]} = \frac{BD}{DC}$). As for (2), the homothety with centre $D$ transforming $B$ into $C$ transforms $B'$ into $C'$, hence $\overrightarrow{BB'} = \overrightarrow{CC'}$. Similarly, $\frac{EC}{EA} = \frac{CC'}{AA'}$ and $\frac{FA}{FB} = \frac{AA'}{BB'}$ and (2) follows with the help of (1).

Another solution to a 2006 problem

The equalities (1) directly give Menelaus’s relation for the transversal $\ell$: indeed, $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -\frac{v}{w} \cdot \frac{w}{u} \cdot \frac{u}{v} = -1$. Not surprisingly, the barycentric
equation of a line can be a shortcut avoiding the use of Menelaus’s theorem. For example, consider Virgil Nicula’s problem 3156 ([2006 : 305 ; 2007 : 312]):

Let \( \Gamma \) be the circumcircle of \( \Delta ABC \). Let \( M \) be an interior point on the side \( AB \), and let \( N \) be an interior point on the side \( AC \). Let \( D \) be an intersection point of \( MN \) with \( \Gamma \). Prove that

\[
\begin{vmatrix}
MB & AC & NC & AB \\
MA & DB & NA & DC
\end{vmatrix}
= \frac{BC}{DA}.
\]

(3)

In the featured solution, Peter Y. Woo applies Menelaus’s theorem twice. Here is a shorter proof: From (1), the equation of the line \( MN \) can be written as \( x = \frac{MB}{MA} y + \frac{NC}{NA} z \) (not signed distances). Since the signed areas \([DCA]\) and \([DAB]\) are of opposite signs, we obtain \([DBC] = \left| \frac{MB}{MA} [DCA] - \frac{NC}{NA} [DAB] \right| \) if areas are no longer signed. Because \( A, B, C, D \) are concylic, we have \( \sin A = \sin \angle BDC, \sin B = \sin \angle CDA, \sin C = \sin \angle ADB \). It follows that \( DB \cdot DC \sin A = \frac{MB}{MA} DA \cdot DC \sin B - \frac{NC}{NA} DA \cdot DB \sin C \) which, using the proportionality of \( BC, CA, AB \) and \( \sin A, \sin B, \sin C \), easily leads to (3).

A property of the tangents to the circumcircle

To illustrate (2), we prove the following:

Let \( \ell \) be a tangent to the circumcircle \( \Gamma \) of \( \Delta ABC \) and let \( BC = a, CA = b, AB = c, d_a = d(A, \ell), d_b = d(B, \ell), d_c = d(C, \ell) \). Then, one of the numbers \( a \sqrt{d_a}, b \sqrt{d_b}, c \sqrt{d_c} \) is the sum of the other two.

**Proof.** Since the equation of \( \Gamma \) is \( a^2 y z + b^2 z x + c^2 x y = 0 \), the equation of \( \ell \) is \( x(b^2 z_0 + c^2 y_0) + y(c^2 x_0 + a^2 z_0) + z(a^2 y_0 + b^2 x_0) = 0 \) where \((x_0, y_0, z_0)\) are the coordinates of the point of tangency. Expressing that the coefficients of \( x, y, z \) are proportional to \( d_a, d_b, d_c \) (from (2)) and solving for \( x_0, y_0, z_0 \) give

\[
x_0 : y_0 : z_0 = a^2(c^2 d_c + b^2 d_b - a^2 d_a) : b^2(a^2 d_a + c^2 d_c - b^2 d_b) : c^2(b^2 d_b + a^2 d_a - c^2 d_c).
\]

Now, \( d_a x_0 + d_b y_0 + d_c z_0 = 0 \) leads to

\[
a^4 d_a^2 + b^4 d_b^2 + c^4 d_c^2 = 2a^2 b^2 d_a d_b + 2b^2 c^2 d_b d_c + 2c^2 a^2 d_c d_a,
\]

that is,

\[
(a \sqrt{d_a} + b \sqrt{d_b} + c \sqrt{d_c})(a \sqrt{d_a} + b \sqrt{d_b} - c \sqrt{d_c})
\times (b \sqrt{d_b} + c \sqrt{d_c} - a \sqrt{d_a})(c \sqrt{d_c} + a \sqrt{d_a} - b \sqrt{d_b}) = 0
\]

and the result follows. A synthetic proof of this (perhaps new) property would be interesting.

An exercise

To conclude, we propose the following problem to the reader: Let \( E \) and \( F \) be points on the sides \( AC \) and \( AB \), respectively. Show that \([PBC]\) is the geometric mean of \([PAB]\) and \([PCA]\) for some point \( P \) on the line segment \( EF \) if and only if \( AE \cdot AF \geq 4 CE \cdot BF \).