On a Problem Concerning Two Conics

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Abstract

The paper describes a connection between concyclity of four distinct common points of two affine conics and their axes of symmetry.

1 Introduction

The aim of this paper is to solve the following problem:

On a plane we have two conics with four distinct common points. Does a necessary and sufficient condition on interrelations of conics exist for the common points to be concyclic?

We show that the answer to our problem is in connection with an axis of symmetry of a conic and has the following form:

If conics $\mathcal{C}_1$ and $\mathcal{C}_2$ have four distinct common points then these points are concyclic if and only if for an arbitrary axis of symmetry s_1 of $\mathcal{C}_1$ there exists an axis of symmetry s_2 of $\mathcal{C}_2$ such that s_1 is perpendicular (or parallel) to s_2.

The main step in the proof is Theorem 1, which is a criterion for points on a conic to be concyclic. Then we use this theorem in the case where a given conic and a circle have exactly three common points. Here we show how to construct a point C on the conic such that, given two points A, B on the conic, a circle through these points has the same tangent with the conic at C.

2 Preliminaries

The following lemma, which is a vector form of the well known *power of a point theorem* [2, p. 28] is needed.

Lemma 1 *Let A, B, C, D be four distinct points. Let lines AC and BD intersect at X. Then A, B, C, D are concyclic if and only if*

$$\overrightarrow{XA} \cdot \overrightarrow{XC} = \overrightarrow{XB} \cdot \overrightarrow{XD}.$$

To begin with, let us introduce some notation. We say that the perpendicular lines p_1, p_2 form ***a perpendicular lines system*** (or just ***a system***) with a notation $\{p_1, p_2\}$. If there are two systems $\{p_1, p_2\}$ and $\{q_1, q_2\}$ with a property that $p_1 \parallel q_1$ and $p_2 \parallel q_2$ they are called ***congruent*** and are denoted as $\{p_1, p_2\} \cong$

$\{q_1, q_2\}$. If congruent systems have the same intersection point, the notation will be $\equiv$. On the plane equipped with the coordinate system $(x, y) \in \mathbb{R}^2$, we have $\{x, y\}$ where x stands for the x-axis and y stands for the y-axis. We introduce a special system called ***an angle bisectors system*** of non-parallel lines p_1, p_2 denoted as $[p_1, p_2]$. Therefore $[p_1, p_2] = \{s_1, s_2\}$, where s_1 and s_2 are angle bisectors of the lines p_1, p_2. In connection with this system we have the following

Lemma 2 *Let p_1 and p_2 be non-parallel lines with the slopes k_1 and k_2. Then $k_1 = -k_2$ if and only if $\{x, y\} \cong [p_1, p_2]$.*

Proof. Let p_1 and p_2 intersect at $X(x_0, y_0)$. Set $A(x_0 + 1, y_1) \in p_1$ and $B(x_0 + 1, y_2) \in p_2$. From the definition of the slope of the line we get $k_1 = y_1 - y_0$ and $k_2 = y_2 - y_0$. Since $k_1 = -k_2$ if and only if $y_0 = (y_1 + y_2)/2$, we conclude that $k_1 = -k_2$ if and only if the angle bisector of $\angle AXB$ is the same as the perpendicular bisector of AB, which is parallel to the x-axis. ∎

Because we will work with *non-degenerate conics* (*an ellipse*, *a parabola* and *a hyperbola*) we simply call them *conics*. By $\mathcal{C}$ we denote an element of the set of all conics that are not circles. We know that an ellipse and a hyperbola have two perpendicular axes of symmetry s_1 and s_2 so they can be considered a system $\{s_1, s_2\}$. The line perpendicular to s_1 with the common point at the vertex of a parabola is taken as another "axis of symmetry". If $\mathcal{C}$ is an arbitrary conic different from a circle, the corresponding ***axes of symmetry system*** is denoted by $\mathrm{Sim}(\mathcal{C})$.

3 The case of a conic and a circle

The following theorem brings up the connection between concyclity of four distinct points on a conic, their angle bisectors system and axes of symmetry system. What is noteworthy here is that the given four distinct points on a conic can always be given names A, B, C, D so that $AC \nparallel BD$.

Theorem 1 *Let A, B, C, D be four distinct points on $\mathcal{C}$. Then A, B, C, D are concyclic if and only if $[AC, BD] \cong \mathrm{Sim}(\mathcal{C})$, that is, the axes of $\mathcal{C}$ are parallel to the bisectors of the angles formed by the lines AC and BD.*

Proof. Set $\mathrm{Sim}(\mathcal{C}) \equiv \{x, y\}$. Then the equation of $\mathcal{C}$ may be written in the form $y^2 = \alpha x^2 + \beta x + \gamma$ with $\alpha \neq -1$. Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3), D(x_4, y_4)$ be four distinct points on $\mathcal{C}$ and let $X(x_0, y_0) := AC \cap BD$. Let the equation of the line AC be $y_0 - y = k_1(x_0 - x)$ and the equation of the line BD be $y_0 - y = k_2(x_0 - x)$. Then

$$\overrightarrow{XA} \cdot \overrightarrow{XC} = (x_0 - x_1)(x_0 - x_3)(1 + k_1^2),$$
$$\overrightarrow{XB} \cdot \overrightarrow{XD} = (x_0 - x_2)(x_0 - x_4)(1 + k_2^2).$$

Because $y = y_0 - k_1(x_0 - x)$ and $y^2 = \alpha x^2 + \beta x + \gamma$ we obtain the quadratic equation

$$(k_1^2 - \alpha)x^2 + (2k_1y_0 - 2k_1^2x_0 - \beta)x + (k_1^2x_0^2 - 2k_1x_0y_0 + y_0^2 - \gamma) = 0 \qquad (1)$$

with solutions x_1 and x_3. By applying Vieta's formulas [3] we obtain

$$(x_0 - x_1)(x_0 - x_3) = -\frac{\alpha x_0^2 + \beta x_0 + \gamma - y_0^2}{k_1^2 - \alpha}. \qquad (2)$$

Similarly, replacing k_1 with k_2 in (1) yields a quadratic equation with solutions x_2 and x_4 and we obtain the formula for $(x_0 - x_2)(x_0 - x_4)$ after replacing k_1 with k_2 in (2). We always have $k_1^2 \neq \alpha$ and $k_2^2 \neq \alpha$ because otherwise $A = C$ and $B = D$.

Since $X \notin \mathcal{C}$ we have $\alpha x_0^2 + \beta x_0 + \gamma - y_0^2 \neq 0$ and since $\alpha \neq -1$ we can conclude $\overrightarrow{XA} \cdot \overrightarrow{XC} = \overrightarrow{XB} \cdot \overrightarrow{XD}$ if and only if $|k_1| = |k_2|$. We cannot have $k_1 = k_2$ and the only possibility is $k_1 = -k_2$. Now the theorem follows from Lemma 1 and Lemma 2.

In this proof, we made the assumption that k_1 and k_2 existed. But k_1 does not exist if and only if $x_1 = x_3$, so it remains to be proven that $x_1 = x_3$ implies that points A, B, C, D are not concyclic. When $x_1 = x_3$, $\alpha x_0^2 + \beta x_0 + \gamma = y_1^2 = y_3^2$ and therefore $\overrightarrow{XA} \cdot \overrightarrow{XC} = y_0^2 - y_1^2$ and $\overrightarrow{XB} \cdot \overrightarrow{XD} = (1 + k_2^2)(y_0^2 - y_1^2)/(k_2^2 - \alpha)$. Since $\alpha \neq -1$, the four points are not concyclic. ■

In the case of an ellipse $\mathcal{E}$, we can find another proof of Theorem 1 using *an affine transformation*. In what follows, the proof will only be briefly described, whilst the details are left to the reader. Following [1], every function $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form $t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is an invertible 2×2 matrix and $\mathbf{b} \in \mathbb{R}^2$, is said to be an affine transformation of $\mathbb{R}^2$. Two of the properties of t are: it preserves the collinearity of points and it preserves the ratios of lengths along a given straight line. If $\mathrm{Sim}(\mathcal{E}) \equiv \{x, y\}$, then the equation of an ellipse is $(x/a)^2 + (y/b)^2 = 1$. Applying an affine transformation

$$t(\mathbf{x}) = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

we get $t(\mathcal{E}) = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1\}$. Let A, B, C, D be four distinct points on $\mathcal{E}$ and let $X := AC \cap BD$. Then $\lambda, \mu \in \mathbb{R}$ such that $\overrightarrow{XC} = \lambda \cdot \overrightarrow{XA}$ and $\overrightarrow{XD} = \mu \cdot \overrightarrow{XB}$ and therefore $\overrightarrow{XA} \cdot \overrightarrow{XC} = \overrightarrow{XB} \cdot \overrightarrow{XD}$ if and only if $(|XB|/|XA|)^2 = \lambda/\mu$. Let $A' = t(A), B' = t(B), C' = t(C), D' = t(D), X' = t(X)$. From Lemma 1 we get $(|X'B'|/|X'A'|)^2 = \lambda/\mu$. After combining and reducing these two equations in coordinate form, we apply Lemma 2.

We say that two intersecting smooth curves have ***a point of tangency*** if the tangent at that point on the first curve is the same as on the second. The following corollary deals with the case where a conic and a circle have exactly three distinct common points. In that case there exists only one point of tangency (for example in Figure 1b such point is C).

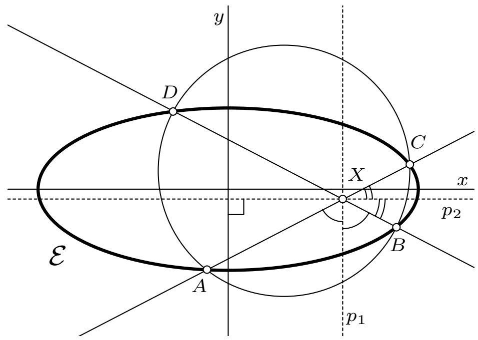

Figure 1a: Example in the sense of Theorem 1 on an ellipse $\mathcal{E}$.

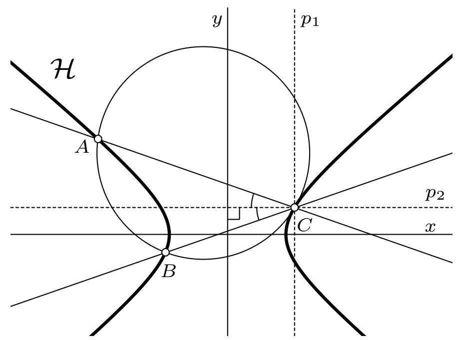

Figure 1b: Example in the sense of Corollary 1 on a hyperbola $\mathcal{H}$.

Given three non-collinear points A, B, C we denote by $\mathcal{K}[A, B, C]$ the circle through these points.

Corollary 1 *Let A, B, C be three distinct points on $\mathcal{C}$. Then C is the point of tangency of $\mathcal{K}[A, B, C]$ and $\mathcal{C}$ if and only if $[AC, BC] \cong \mathrm{Sim}(\mathcal{C})$.*

Proof. Let $[AC, BC] \cong \mathrm{Sim}(\mathcal{C})$. Denote by $B_\varepsilon(C)$ an open Euclidean ball with the radius $\varepsilon > 0$ at a point C. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive real numbers, such that $\lim_{n\to\infty} \varepsilon_n = 0$. For each $n \in \mathbb{N}$ points $C'_n, C''_n \in B_{\varepsilon_n}(C) \cap \mathcal{C}$ exist with $[AC'_n, BC''_n] \cong \mathrm{Sim}(\mathcal{C})$ and therefore A, B, C'_n, C''_n are concyclic by Theorem 1. Because the boundary of $\mathcal{C}$ is smooth, the sequence of the lines $C'_nC''_n$ converge to the tangent of $\mathcal{C}$ at C. The limit circle is therefore $\mathcal{K}[A, B, C]$ which has a common tangent with $\mathcal{C}$ at C.

On the other hand, let $[AC, BC] \ncong \mathrm{Sim}(\mathcal{C})$. Then a point $X \in AC$, $X \neq C$ exists such that $[AC, BX] \cong \mathrm{Sim}(\mathcal{C})$. If XB is not a tangent on $\mathcal{C}$ and $X \neq A$, then there exists a point $D \in \mathcal{C} \cap BX$. By Theorem 1 it follows that A, B, C, D are concyclic and therefore C is not the point of tangency of $\mathcal{K}[A, B, C]$ and $\mathcal{C}$. If $X = A$, then A is the point of tangency by the first part of this proof, so C is not. If XB is a tangent, then for each $n \in \mathbb{N}$ points $B'_n, B''_n \in B_{\varepsilon_n}(B) \cap \mathcal{C}$ exist with $B'_nB''_n \parallel XB$ and therefore A, C, B'_n, B''_n are concyclic by Theorem 1. We conclude that B is the point of tangency of $\mathcal{K}[A, B, C]$ and $\mathcal{C}$. In any case C cannot be the point of tangency, this completes the proof. ∎

With the same method as in the proof of Corollary 1 we are able to prove another result concerning points of tangency of a circle and a conic:

> *Let A, B, C be three distinct points on $\mathcal{C}$. Then C is the point of tangency of $\mathcal{K}[A, B, C]$ and $\mathcal{C}$ if and only if $[AB, t] \cong \mathrm{Sim}(\mathcal{C})$ where t is a tangent on $\mathcal{C}$ at C.*

Given two points A, B on a conic $\mathcal{C}$, in general there are two different circles $\mathcal{K}_1$ and $\mathcal{K}_2$ through A, B where A and B are consecutive points of tangency. But a point $C \in \mathcal{C}$ exists such that C is the point of tangency of $\mathcal{K}[A,B,C]$ and $\mathcal{C}$. Define $C_1 \in \mathcal{K}_1 \cap \mathcal{C}$, $C_2 \in \mathcal{K}_2 \cap \mathcal{C}$ and t as a tangent on $\mathcal{C}$ at C (see Figure 2). Combining Corollary 1 and the above result we obtain $[BA, C_1A] \cong [AB, C_2B] \cong [AB, t]$ and therefore $AC_1 \parallel BC_2 \parallel t$. Using the *midpoint theorem for conics: given chord l of a conic, then the midpoints of the chords parallel to l are collinear* (see [1] for the proof in the case of an ellipse and a hyperbola) we are able to construct C as the point of tangency from the given points A and B. In Figure 2 there are two parallel chords AC_1 and BC_2 with consecutive midpoints M_1 and M_2. Then $C = M_1M_2 \cap \mathcal{P}$. When a conic is an ellipse or a hyperbola, the midpoint theorem further ensures that the line through midpoints of the chords goes through the center of a conic. In that case there exist two points $C_1, C_2 \in \mathcal{C}$ such that C_i is the point of tangency of $\mathcal{K}[A,B,C_i]$ and $\mathcal{C}$ for $i \in \{1,2\}$. The reader is invited to draw some figures of these cases.

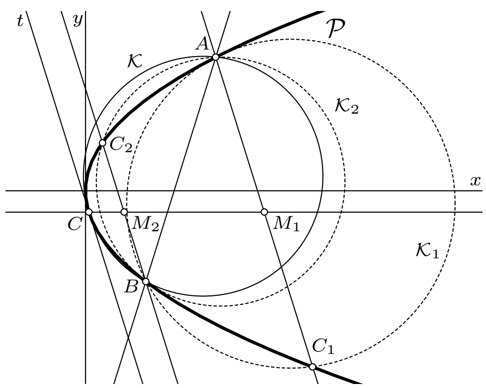

Figure 2: A parabola $\mathcal{P}$ with circles $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}$ and consecutive points of tangency A, B, C.

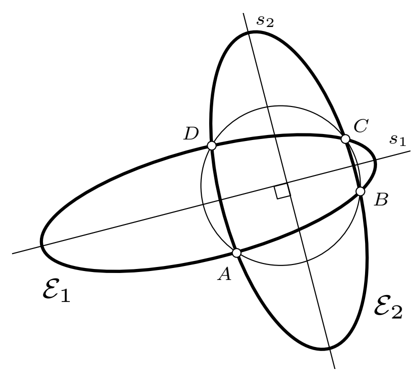

Figure 3: Two ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$ with four concyclic common points A, B, C, D.

4 Solution

In this last section we will provide an answer to our problem. We need the following fact about conics: if $|\mathcal{C}_1 \cap \mathcal{C}_2| = 4$, then the conics $\mathcal{C}_1$ and $\mathcal{C}_2$ do not have a point of tangency.

Corollary 2 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two conics with four distinct common points. These points are concyclic if and only if $\mathrm{Sim}(\mathcal{C}_1) \cong \mathrm{Sim}(\mathcal{C}_2)$.*

Proof. Let the common points be A, B, C, D. If they are concyclic, we have $\mathrm{Sim}(\mathcal{E}_1) \cong [AC, BD] \cong \mathrm{Sim}(\mathcal{E}_2)$.

Suppose $\mathrm{Sim}(\mathcal{C}_1) \cong \mathrm{Sim}(\mathcal{C}_2)$ and assume (for a contradiction) that A, B, C, D are not concyclic. Denote $\mathcal{K} := \mathcal{K}[A,B,C]$ and since points are not concyclic $D \notin \mathcal{K}$. There are three possibilities: $|\mathcal{K} \cap \mathcal{C}_1| = 3$ and $|\mathcal{K} \cap \mathcal{C}_2| = 3$, $|\mathcal{K} \cap \mathcal{C}_1| = 3$ or $|\mathcal{K} \cap \mathcal{C}_2| = 3$, $|\mathcal{K} \cap \mathcal{C}_1| = 4$ and $|\mathcal{K} \cap \mathcal{C}_2| = 4$. In the first case two points of tangency among A, B, C, say A and B, exist. From Corollary 1 we get $[BA, CA] \cong \mathrm{Sim}(\mathcal{C}_1)$

and $[AB, CB] \cong \mathrm{Sim}(\mathcal{C}_2)$. But then $[BA, CA] \cong [AB, CB]$, which is a contradiction. In the second case let us have $|\mathcal{K} \cap \mathcal{C}_1| = 3$ with C as the point of tangency and $|\mathcal{K} \cap \mathcal{C}_2| = 4$. Define $D \in \mathcal{K} \cap \mathcal{C}_2$. We assume without loss of generality that $AD \nparallel BC$. But then this is a contradiction with Theorem 1 and Corollary 1 because we get $AD \parallel AC$. In the third case define $D_1 \in \mathcal{K} \cap \mathcal{C}_1$ and $D_2 \in \mathcal{K} \cap \mathcal{C}_2$. Then $D_1 \neq D_2$ but from Theorem 1 we get $BD_1 \parallel BD_2$ which is a contradiction. ■

It is clear that the answer from the introduction of this paper is equivalent to Corollary 2. For example, in Figure 3 we have two ellipses with four concyclic common points. We observe that s_1 is perpendicular to s_2 where s_1 is an axis of symmetry of $\mathcal{E}_1$ and s_2 is an axis of symmetry of $\mathcal{E}_2$. This agrees with our findings.

References

[1] D. A. Brannan, M. F. Esplen, J. J. Gray, *Geometry*, Cambridge University Press, Cambridge, 1999.

[2] H. S. M. Coxeter, S. L. Greitzer, *Geometry revisited*, New Mathematical Library 19, MAA, 1967.

[3] E. W. Weisstein, *Vieta's Formulas*, MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/VietasFormulas.html`.

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