SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

210. [1977 : 10, 160-164, 196-198; 1978 : 13-16, 193-194] *Proposed by the late Murray S. Klamkin, University of Alberta, Edmonton, AB.*

P, Q, R denote points on the sides BC, CA, and AB, respectively, of a given triangle ABC. Determine all triangles ABC such that if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \quad \left(\neq 0, \frac{1}{2}, 1\right) \tag{1}$$

then PQR (in some order) is similar to ABC.

VII. Comment by Grégoire Nicollier, University of Applied Sciences of Western Switzerland, Sion, Switzerland.

Previous discussions of this problem have dealt with a fixed triangle ABC and determined those values of k for which there exists an inscribed triangle PQR that is similar to it. As stated, however, problem 210 calls for the converse: Given a real number k, determine ΔABC with the desired property. It turns out that there is a family of solution triangles for every value of k. Before making that precise, let us review the facts that have already been established.

First, note that in the excluded cases ($k = 0, 1, \frac{1}{2}$), for all triangles ABC the triangle PQR would be BCA, CAB, and the medial triangle of ABC, respectively, which are always directly similar to ABC. Of course, when ABC is equilateral, every real value of k produces an equilateral triangle PQR; conversely, solutions II [1977 : 163] and III [1977 : 197-198] proved that if ΔPQR is *directly* similar to ΔABC and $k \neq 0, 1, \frac{1}{2}$, then those triangles are necessarily equilateral. In general (see [1]),

(a) if one excepts the equilateral case, the solutions k exist only for scalene triangles ABC, and these solutions are the signed ratios

$$k_1 = \frac{a^2 - b^2}{2a^2 - b^2 - c^2}, \quad k_2 = \frac{b^2 - c^2}{2b^2 - c^2 - a^2}, \quad k_3 = \frac{c^2 - a^2}{2c^2 - a^2 - b^2}. \tag{2}$$

The similarity ratio is then $\sqrt{1 - 3k_\ell + 3k_\ell^2}$ and the similarity is always opposite with $PQ : QR : RP = b : a : c$ for k_1, $a : c : b$ for k_2, and $c : b : a$ for k_3 (these formulae are incorrect in [1] and in [2]).

For the present paragraph only, we adopt the convention that our triangle ABC has been labeled so that $a > b > c$. Triangles for which $2b^2 = c^2 + a^2$ (in which case the denominator of k_2 is zero) were called *Root-Mean-Square* triangles in [1978 : 14-16; 2010 : 304-307], but Europeans seem to prefer the terminology *automedian* (since these triangles are characterized by being similar to the triangle formed by the three medians, a triangle that is sometimes called the *median triangle* [1978 : 14]). Triangle ABC with $a > b > c$ is automedian if and only if the vertex opposite to the middle side b lies on the circle of radius $\frac{\sqrt{3}}{2}b$ centered at the midpoint of b (as does the apex of the equilateral triangle erected on b) [1978 : 13]. Continuing with the results from [1],

(b) one has $k_{\ell+1} = \frac{1-2k_\ell}{1-3k_\ell}$ for $\ell = 1, 2, 3$ by setting $k_4 = k_1$. With sides labeled so that $a > b > c$, we have $0 < k_1 < \frac{1}{2} < k_3 < 1$, while the value k_2 lies outside $[0, 1]$, except that it fails to exist when ΔABC is automedian.

An alternative approach to these and related results that combines the discrete Fourier transformation, convolution products, and a shape function for triangles, can be found in [3]. We turn now to the promised solution to problem 210.

We first look at the behavior of PQR for $k \to \pm\infty$: The angles of PQR tend to the angles of the triangle whose vertices are the tips of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$ when their common origin is the centroid of ABC. A homothety of ratio $1/2$ about the centroid transforms this triangle into a triangle that is directly congruent to that formed by the medians of ABC. The triangle PQR is thus directly similar to the median triangle of ABC in the three cases $k = \frac{1}{3}$, $k = \frac{2}{3}$, and $k = \infty$ (the point at infinity of the extended real line). We can therefore assert that automedian scalene triangles also have three solutions: $k = \frac{1}{3}$, $\frac{2}{3}$, and ∞.

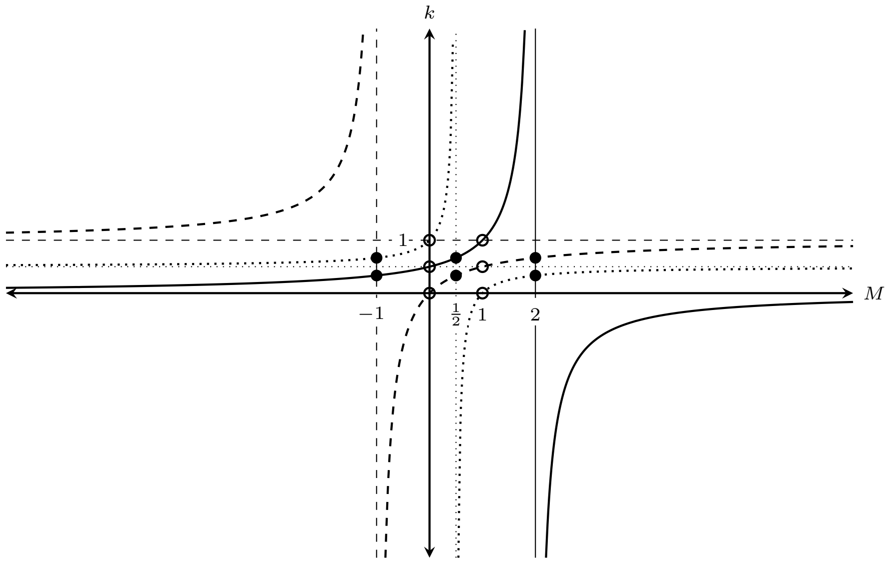
From now on, the symbols A, B, and C will denote complex numbers as well as the vertices of the triangle that they represent. Because we will not require multiplication, we will continue to denote our triangles by ABC (which is not a product!). The *normalized triangle* $\Delta(z)$ corresponding to ΔABC is a triangle with vertices 0, 1, and $z := \frac{C-A}{B-A}$. It is the unique normalized triangle directly similar to ABC with 0 corresponding to A, 1 to B, and z to C.

Theorem 1 *When ΔABC is scalene with normalized triangle $\Delta(z)$, the circle through $z = \frac{C-A}{B-A}$, $e^{i\pi/3}$, and $e^{-i\pi/3}$ is centered at $M_1 = \frac{b^2-c^2}{b^2-a^2}$. BCA corresponds to a normalized triangle with center $M_2 = \frac{c^2-a^2}{c^2-b^2}$, and CAB to $M_3 = \frac{a^2-b^2}{a^2-c^2}$.*

Proof. Note that $e^{\pm i\pi/3}$ are the apices of the normalized equilateral triangles. With $z = x + iy$, the center M of the circle corresponding to ABC is given by $(x - M)^2 + y^2 = r^2 = M^2 - M + 1$, i.e., $M = \frac{x^2+y^2-1}{2x-1} = M_1$. ∎

The solutions k_ℓ given by (2) are cyclically related by $k \mapsto \frac{1-2k}{1-3k}$, and the three circle centers M_ℓ by $M \mapsto 1 - \frac{1}{M}$. Note that $M_1M_2M_3 = -1$. The center M_ℓ beginning (in Theorem 1) with the middle side is negative. If one has $a > b > c$, $b > c > a$, or $c > a > b$, the center M_ℓ beginning with the longest side lies between 0 and 1, and the center beginning with the shortest side is greater than 1. The situation is reversed in the other arrangements of a, b, c.

The points (M_1, k_1), (M_2, k_2), (M_3, k_3) lie in the accompanying figure on the plain hyperbola $k = f_1(M) = \frac{1}{2-M}$, i.e., $M = 2 - \frac{1}{k}$. The points (M_1, k_2),



(M_2, k_3), (M_3, k_1) lie on the dashed hyperbola $k = f_2(M) = \frac{M}{M+1}$, i.e., $M = \frac{k}{1-k}$. And the points (M_1, k_3), (M_2, k_1), (M_3, k_2) lie on the dotted hyperbola $k = f_3(M) = \frac{1-M}{1-2M}$, i.e., $M = \frac{1-k}{1-2k}$. These three hyperbolas are cyclically related by $f_{\ell+1}(M) = f_\ell\Big(1 - \frac{1}{M}\Big)$.

The six sets

$$\{f_1(M_\ell), f_2(M_\ell), f_3(M_\ell)\} \quad \text{and} \quad \{f_\ell(M_1), f_\ell(M_2), f_\ell(M_3)\}, \quad \ell = 1, 2, 3,$$

are all equal to the solution set $\{k_1, k_2, k_3\}$. One can retrieve all solutions k_ℓ from a single center M', and all circle centers M_ℓ from a single solution k' by intersecting the three hyperbolas with the lines $M = M'$ and $k = k'$. When the triangle is not automedian, one circle center lies between $\frac{1}{2}$ and 2, one between -1 and $\frac{1}{2}$, and one outsides $[-1, 2]$. Note that $M = -1$, $\frac{1}{2}$, 2 correspond to normalized automedian triangles and $M = 0$, 1, ∞ to isosceles ones.

Theorem 2 *The scalene ΔABC is a k-triangle (i.e., ΔABC is similar to ΔPQR given by* (1) *for that value of k) if and only if the circle through $e^{\pm i\pi/3}$ and the apex of the normalized triangle of ABC is centered at $2 - \frac{1}{k}$, $\frac{k}{1-k}$, or $\frac{1-k}{1-2k}$.*

Proof. The given circle is centered at $M_1 = \frac{b^2-c^2}{b^2-a^2}$ by Theorem 1. By equation (2), $M_1 = 2 - \frac{1}{k}$, $\frac{k}{1-k}$, and $\frac{1-k}{1-2k}$ if and only if $k = k_1$, k_2, and k_3, respectively. ■

Theorem 2 can also be formulated as follows: the scalene ΔABC is a k-triangle if and only if the circle through $e^{i\pi/3}$ centered at $2 - \frac{1}{k}$ contains the apex of a normalized triangle similar (in some order) to ABC. One can replace $2 - \frac{1}{k}$ by $\frac{k}{1-k}$ and by $\frac{1-k}{1-2k}$.

References

[1] W. J. Blundon and D. Sokolowsky, Solution to Problem 210 (Klamkin), *Crux Math.* 3 (1977) 160–164 and 196-197.

[2] A. Liu and B. Shawyer (eds.), *Problems from Murray Klamkin: The Canadian Collection*, The Mathematical Association of America, Washington, DC, 2008.

[3] G. Nicollier, Convolution filters for triangles, *Forum Geom.* 13 (2013) 61–85. http://forumgeom.fau.edu/FG2013volume13/FG201308index.html

3653. [2011 : 318, 321; 2012 : 247] *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let O be the centre of a sphere S circumscribing a tetrahedron $ABCD$. Prove that:

(i) there exists tetrahedra whose four faces are obtuse triangles; and

(ii) ⋆ if O is inside or on $ABCD$, then at least two faces of $ABCD$ are acute triangles.

II. Combination of solutions by Tomasz Cieśla, student, University of Warsaw, Poland; and the proposer.

The statement of part (ii) is not quite correct. To repair it, we will revise both statements. In the solution to the new part (i) we will extend the solution that appeared before in [2012 : 247].

(i) (Revised) *If O is outside $ABCD$ or at the midpoint of one of its edges, then it is possible that no faces are acute triangles.*

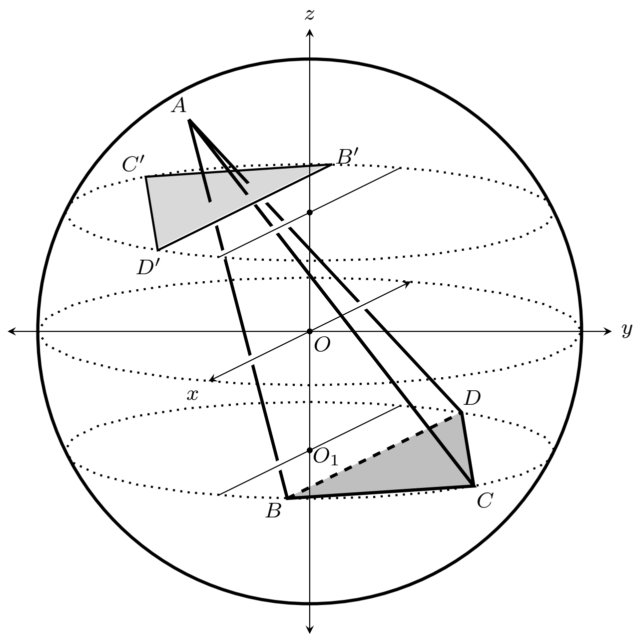
Start with a trapezoid $ABCD$ for which $AD||BC$ and the angles $\angle BAC$, $\angle BDC$ are both obtuse. Then triangles BAD and CDA are also obtuse. Now lift B a short distance above the plane so that lines AC and BD are skew. Then $ABCD$ is no longer a trapezoid, but a tetrahedron whose four faces are obtuse triangles. The location of its circumcentre will be a consequence of the argument in part (ii). For an example that has O on an edge, take the convex hull of the diagonal of a cube and a disjoint edge (for example, the tetrahedron with vertices $(0,0,0),(1,0,0),(1,1,0),(0,0,1)$). All four of its faces will be right triangles.

(ii) (Revised) *If O is inside the tetrahedron $ABCD$ or in the interior of one of its faces, then at least two faces are acute triangles.*

For our proof we require two lemmas that follow immediately from the corresponding plane theorems.

Lemma 1. For three points P,Q,R on a sphere with centre O, the plane triangle PQR is acute if and only if the foot O' of the perpendicular from O to the plane lies inside the triangle. (Note that O' is the circumcentre of ΔPQR, so that this result reduces to the familiar plane theorem.)

Lemma 2. For three points PQR on a sphere with centre O, the plane containing the segment QR and perpendicular to the line joining O to its midpoint will intersect the sphere in a circle whose diameter is QR; $\angle QPR$ will be right, acute, or obtuse according as the point P lies on that circle, in the larger cap determined by that circle, or in the smaller cap.



We turn now to the proof of (ii). We are given $ABCD$ inscribed in a sphere whose centre lies inside the tetrahedron or in the interior of one of its faces; we are to prove that at least two of its four faces are acute triangles. To this end we assume that $\angle BCD$ is an obtuse or right angle and will show that there can be at most one further angle that is not acute. In the accompanying figure we display the circumsphere with its centre at the origin of a cartesian coordinate system, with the x-axis parallel to BD, and the y- and z-axes oriented so that the xy-plane is parallel to and above the plane of the face BCD. By Lemma 1, the projection O_1 of O into the plane of BCD must lie outside the triangle or on the edge BD. Moreover, the line AO must strike that plane inside the triangle or at the midpoint of BD. Reflecting that triangle in O to $\Delta B'C'D'$, we see that A will be in the portion of the sphere that is separated from ΔBCD by the vertical and horizontal planes through $B'D'$. Consequently, A lies in the larger cap determined by the spherical circle on the diameter DB. By Lemma 2 $\angle DAB$ must be acute. Next, observe that A lies with D in the cap determined by the circle on diameter BD'; because $\angle DBX = 90^\circ$ for all points X on that circle, we deduce that $\angle DBA$ is acute. The same argument with the roles of B and D interchanged shows that $\angle BDA$ is also acute; in summary,

$$\angle DAB, \angle DBA, \text{ and } \angle BDA \text{ are all acute angles.}$$

In other words, we have shown that if a face of our tetrahedron has a nonacute angle at one vertex, the face of the tetrahedron opposite that vertex must be an acute triangle. It follows that for our tetrahedron to have three nonacute face angles, they would necessarily have C as a vertex. But it is easily seen that the circumcentre of a tetrahedron having three nonacute angles at C would necessarily be exterior to the tetrahedron: Returning to the figure, we see that the line through C perpendicular to the plane BCD (and therefore parallel to OO_1) would intersect the sphere at a point, call it P that is separated from the vertex A by the plane through BD that is parallel to CP. The faces BCP and DCP of the tetrahedron $PBCD$ have right angles at P. To make those angles obtuse, P would have to be chosen on the sphere further from A; that is, at least one of the angles BCA and DCA would have to be acute.

3671★. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $ABCD$ be a tetrahedron and let M be a point in its interior. Prove or disprove that

$$\frac{[BCD]}{AM^2} = \frac{[ACD]}{BM^2} = \frac{[ABD]}{CM^2} = \frac{[ABC]}{DM^2} = \frac{2}{\sqrt{3}},$$

if and only if the tetrahedron is regular and M is its centroid. Here $[T]$ denotes the area of T.

No solutions have been received. This problem remains open.

3672. [2011 : 389, 392] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x and y be real numbers such that $x^2 + y^2 = 1$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+xy} \;\geq\; \frac{3}{1+\left(\frac{x+y}{2}\right)^2}.$$

When does this inequality occur?

Solution by Arkady Alt, San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Marian Dincă, Bucharest, Romania; Dimitrios Koukakis, Kato Apostoloi, Greece; Kee-Wai Lau, Hong Kong, China; Salem Malikić, student, Simon Fraser University, Burnaby, BC; Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA; Madhav R. Modak, formerly of Sir Parashurambhau College, Pune, India; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comǎneşti, Romania; and the proposer.

Let $t = xy$. Since $2|xy| \leq x^2 + y^2 = 1$, $|t| \leq \frac{1}{2}$. The difference between the two sides of the proposed inequality is

$$\begin{aligned}\frac{2+x^2+y^2}{1+x^2+y^2+x^2y^2}+\frac{1}{1+xy}-\frac{12}{4+x^2+y^2+2xy}&\\ =\frac{3}{2+t^2}+\frac{1}{1+t}-\frac{12}{5+2t}&\\ =\frac{(1-2t)(1+3t+5t^2)}{(2+t^2)(1+t)(5+2t)}=\frac{(1-2t)(1+t^2+(1+3t)^2)}{2(2+t^2)(1+t)(5+2t)}&\\ \geq 0&\end{aligned}$$

with equality if and only if t = 1/2. With the given condition, this implies that equality occurs if and only if $x = y = \pm 1/\sqrt{2}$.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; STAN WAGON, Macalester College, St. Paul, MN, USA; and PETER Y. WOO, Biola University, La Mirada, CA, USA;

Wagon used mathematical software to find that, when $x^2+y^2=1$,

$$3 \leq \left(\frac{1}{1+x^2}+\frac{1}{1+y^2}+\frac{1}{1+xy}\right)\left(1+\left(\frac{x+y}{2}\right)^2\right) \leq \frac{10}{3}$$

with equality on the left if and only if $x=y=\pm 1/\sqrt{2}$. *and equality on the right if and only if* $x=-y=\pm 1/\sqrt{2}$.

3673. [2011 : 390, 392] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Calculate the product

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)^{(-1)^{n-1}} .$$

I. Composite of solutions by Paul Bracken, University of Texas, Edinburg, TX, USA; Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA; Dimitrios Koukakis, Kato Apostoloi, Greece; and AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

The answer is $\pi^2/8$. Recall the Wallis formula:

$$\lim_{m\to\infty}\frac{1}{2m+1}\prod_{k=1}^{m}\frac{(2k)^2}{(2k-1)^2}=\frac{\pi}{2}.$$

For $n \geq 2$, let

$$P(n)=\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)^{(-1)^{k-1}} .$$

Then

$$\begin{aligned}
P(2m) &= \prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)(2k+1)}\right]\prod_{k=1}^{m-1}\left[\frac{(2k)(2k+2)}{(2k+1)^2}\right]\\
&= \frac{(2m)^2}{(2m-1)(2m+1)}\prod_{k=1}^{m-1}\left[\frac{(2k)^2}{(2k+1)^2}\right]\prod_{k=1}^{m-1}\left[\frac{2k}{2k-1}\right]\prod_{k=1}^{m-1}\left[\frac{2k+2}{2k+1}\right]\\
&= \frac{(2m)^2}{(2m-1)(2m+1)}\left(\frac{2m}{2m-1}\right)\frac{1}{2}\frac{1}{(2m)^2}\prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)^2}\right]\\
&\qquad\qquad\times\prod_{k=1}^{m-1}\left[\frac{2k}{2k-1}\right]\prod_{k=1}^{m-1}\left[\frac{2k}{2k-1}\right]\\
&= \frac{(2m)^2}{(2m-1)(2m+1)}\left(\frac{2m}{2m-1}\right)\frac{1}{2}\frac{1}{(2m)^2}\left[\frac{(2m-1)^2}{(2m)^2}\right]\\
&\qquad\qquad\times\prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)^2}\right]\prod_{k=1}^{m}\left[\frac{(2k)^2}{(2k-1)^2}\right]\\
&= \frac{2m+1}{2(2m)}\left[\frac{1}{2m+1}\prod_{k=1}^{m}\frac{(2k)^2}{(2k-1)^2}\right]^2,
\end{aligned}$$

and

$$P(2m+1) = \frac{(2m)(2m+2)}{(2m+1)^2}P(m).$$

Hence

$$\begin{aligned}
\lim_{m\to\infty}P(2m+1) &= \lim_{m\to\infty}\frac{(2m)(2m+2)}{(2m+1)^2}\lim_{m\to\infty}P(2m)\\
&= \lim_{m\to\infty}P(2m) = \lim_{m\to\infty}\left[\frac{2m+1}{4m}\right]\frac{\pi^2}{4} = \frac{\pi^2}{8}.
\end{aligned}$$

II. Composite of solutions by Michel Bataille, Rouen, France; Anastasios Kotronis, Athens, Greece; Kee-Wai Lau, Hong Kong, China; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

Recall Stirling's formula:

$$\lim_{n\to\infty}\frac{(e^n)n!}{n^n\sqrt{2\pi n}} = 1.$$

Defining $P(n)$ as in Solution I, we have that

$$\begin{aligned} P(2m+1) &= \frac{1}{2} \cdot \frac{[2 \cdot 4 \cdot 6 \cdots (2m)]^4 (2m+2)(2m+1)}{[3 \cdot 5 \cdot 7 \cdots (2m+1)]^4} \\ &= \frac{[2 \cdot 4 \cdots 6 \cdots (2m)]^8 (m+1)(2m+1)}{[(2m+1)!]^4} \\ &= \frac{2^{8m}(m+1)(m!)^8}{(2m+1)^3[(2m)!]^4}. \end{aligned}$$

Hence

$$\begin{aligned} \lim_{m\to\infty} P(2m+1) &= \lim_{m\to\infty} \frac{2^{8m}(m+1)(2\pi m)^4 m^{8m} e^{-8m}}{(2m+1)^3 (4\pi m)^2 (2m)^{8m} e^{-8m}} \\ &= \lim_{m\to\infty} \frac{(m+1)m^2\pi^2}{(2m+1)^3} = \frac{\pi^2}{8} \end{aligned}$$

and

$$\lim_{m\to\infty} P(2m) = \lim_{m\to\infty} \frac{(2m+1)^2}{(2m)(2m+2)} P(2m+1) = \frac{\pi^2}{8}.$$

III. Composite of solution by Richard I. Hess, Rancho Palos Verdes, CA, USA; Dimitrios Koukakis, Kato Apostoloi, Greece; Missouri State University Problem Solving Group, Springfield, MO; Skidmore College Problem Group, Saratoga Springs, NY; Albert Stadler, Herrliberg, Switzerland; and the proposer.

We use the infinite product representations for the sine and cosine functions:

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right);$$

$$\cos \pi x = (1 - 4x^2) \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n+1)^2}\right).$$

The product of the problem can be written as a fraction whose numerator is

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{(2n+1)^2}\right) = \lim_{x\to\frac{1}{2}} \frac{\cos \pi x}{1 - 4x^2} = \lim_{x\to\frac{1}{2}} \frac{\pi \sin \pi x}{8x} = \frac{\pi}{4},$$

and whose denominator is

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{(2n)^2}\right) = \frac{\sin \pi/2}{\pi/2} = \frac{2}{\pi}.$$

It follows from this that the answer is $\pi^2/8$.

There were variants on the third solution. Stadler and the Skidmore group avoided consideration of the cosine product by writing the product as a fraction with numerator $\prod(1-1/n^2)$ and denominator $\prod(1-1/4n^2)^2$. Koukakis used the Wallis formula instead of the sine product to calculate the denominator.

Stan Wagon generalized the product to

$$\prod_{n=2}^{\infty}\left(1-\frac{a}{n^2}\right)^{(-1)^n},$$

so that $a=1$ gives the reciprocal of the given product. Using mathematical software, he then provided the answer for specific values of a. For example, when $a=2$, the product is $-\sqrt{2}\tan(\pi/\sqrt{2})/\pi$; when $a=1/2$, it is $\sqrt{2}\tan(\pi/2\sqrt{2})/\pi$; when $a=4$, it is 0 and when $a=1/4$, it is $3/\pi$. He concludes that, setting $b=1/a$, "indeed, there seems to be general formula here that looks like

$$\frac{2(b-1)\tan(\pi/2\sqrt{b})}{\pi\sqrt{b}},$$

though I have not investigated the exact range of truth for it." When $a=b=1$, we find through l'Hôpital's Rule that the limit as b tends to 1 is the expected $8/\pi^2$.

Wagon's recourse to mathematical software raises two issues. Many of the Crux problems can be easily handled by machine, but are still worth posing when they can attract solutions that reveal underlying structure or when they draw attention to a particularly comely mathematical fact. The challenge is not to just solve the problem, but to do so in a way that is elegant, interesting or insightful. The efficiency of the software allows for experimentation that leads to conjectures not otherwise obtainable. This problem is a good example of these effects.

The proposer also asked for the values of the infinite products

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)^{(-1)^{n-1}}$$

and

$$\prod_{n=2}^{\infty}\left(\frac{n^2+1}{n^2-1}\right)^{(-1)^{n-1}}.$$

The value of the first is $\frac{\pi}{2}\tanh\frac{\pi}{2}$ and of the second is $\frac{2}{\pi}\tanh\frac{\pi}{2}$.

3674★. *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let I denote the centre of the inscribed sphere of a tetrahedron $ABCD$ and let A_1, B_1, C_1, D_1 denote their symmetric points of point I about planes BCD, ACD, ABD, ABC respectively. Must the four lines AA_1, BB_1, CC_1, DD_1 be concurrent?

No solutions have been received. This problem remains open.

3675. [2011 : 390, 392] *Proposed by Michel Bataille, Rouen, France.*

Let a, b, and c be the sides of a triangle and let s be its semiperimeter. Let r and R denote its inradius and circumradius respectively. Prove that

$$6 \le \sum_{\text{cyclic}}\frac{b(s-b)+c(s-c)}{a(s-a)} \le \frac{3R}{r}.$$

Solution by the proposer.

By regrouping the middle expression to $\sum_{\text{cyclic}}\left(\frac{a(s-a)}{b(s-b)}+\frac{b(s-b)}{a(s-a)}\right)$, the left-hand inequality follows immediately from the fact that $x+\frac{1}{x}\geq 2$ for positive x. The right-hand inequality rewrites as

$$\sum_{\text{cyclic}} bc(s-b)(s-c)\left[b(s-b)+c(s-c)\right]\leq \frac{3R}{r}\cdot abc\cdot(s-a)(s-b)(s-c). \quad (1)$$

Using Heron's formula and the fact that $rs=\frac{abc}{4R}$, the right-hand side of (1) becomes

$$\frac{3R}{r}\cdot abc\cdot\frac{(rs)^2}{s}=3R(abc)(rs)=\frac{3(abc)^2}{4}.$$

Now, since $(s-b)(s-c)\leq\left(\frac{s-b+s-c}{2}\right)^2=\frac{a^2}{4}$ for all cyclic permutations of a, b, and c, then the left-hand side L of (1) satisfies

$$\begin{aligned} L &\leq \frac{abc}{4}\left(\sum_{\text{cyclic}} a\left[b(s-b)+c(s-c)\right]\right)=\frac{abc}{4}\left(\sum_{\text{cyclic}} ab(s-b+s-a)\right)\\ &=\frac{abc}{4}(abc+abc+abc)=\frac{3(abc)^2}{4}. \end{aligned}$$

It follows that (1) holds.

Note that equality (on either side) occurs if and only if the triangle is equilateral and that the result of the problem improves the classical inequality $R\geq 2r$.

Also solved by ARKADY ALT, San Jose, CA, USA; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; DIMITRIOS KOUKAKIS, Kato Apostoloi, Greece; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DRAGOLJUB MILOŠEVIĆ, Gornji Milanovac, Serbia; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and TITU ZVONARU, Comăneşti, Romania.

3677. [2011 : 454, 456] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n-1}(-1)^k\sin^n(k\pi/n)=\frac{(1+(-1)^n)n}{2^n}\cdot\cos\frac{n\pi}{2}.$$

Solution by Anastasios Kotronis, Athens, Greece.

[*Ed.: For the summation to make sense, we assume that* $n\geq 2$.]

Let $w=e^{\frac{i\pi}{n}}=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$. Then $w^n=\cos\pi=-1$, so for all $k=1,2,\ldots,n-1$ we have $w^{kn}=(-1)^k$ and $w^{2kn}=1$.

Since $\sin\frac{k\pi}{n} = \dfrac{w^k - w^{-k}}{2i}$, we have

$$\begin{aligned}
\sum_{k=1}^{n-1}(-1)^k \sin^n\left(\frac{k\pi}{n}\right) &= \sum_{k=1}^{n-1}(-1)^k\left(\frac{w^k - w^{-k}}{2i}\right)^n \\
&= \sum_{k=1}^{n-1}\left(\frac{-i}{2}\right)^n w^{kn}(w^k - w^{-k})^n = \left(\frac{-i}{2}\right)^n \sum_{k=1}^{n-1}(w^{2k}-1)^n \\
&= \left(\frac{-i}{2}\right)^n \sum_{k=1}^{n-1}\left(\sum_{m=0}^{n}(-1)^m\binom{n}{m}w^{2km}\right) \\
&= \left(\frac{-i}{2}\right)^n \sum_{m=0}^{n}\left((-1)^m\binom{n}{m}\sum_{k=1}^{n-1}w^{2km}\right) \\
&= \left(\frac{-i}{2}\right)^n \left(\binom{n}{0}(n-1) + (-1)^n\binom{n}{n}(n-1)\right. \\
&\qquad\qquad \left. + \sum_{m=1}^{n-1}(-1)^m\binom{n}{m}\sum_{k=1}^{n-1}w^{2km}\right) \\
&= \left(\frac{-i}{2}\right)^n \left((1+(-1)^n)(n-1)\right. \\
&\qquad\qquad \left. + \sum_{m=1}^{n-1}(-1)^m\binom{n}{m}\left(\frac{w^{2mn}-w^{2m}}{w^{2m}-1}\right)\right) \\
&= \left(\frac{-i}{2}\right)^n \left((1+(-1)^n)(n-1)\right. \\
&\qquad\qquad \left. - \sum_{m=1}^{n-1}(-1)^m\binom{n}{m}\right),
\end{aligned}$$

since $w^{2mn} = 1$ for $m = 1, 2, \ldots, n-1$. Thus

$$\begin{aligned}
\sum_{k=1}^{n-1}(-1)^k \sin^n\left(\frac{k\pi}{n}\right) &= \left(\frac{-i}{2}\right)^n \left((1+(-1)^n)(n-1)\right. \\
&\qquad\qquad \left. - \left(0 - (-1)^0\binom{n}{0} - (-1)^n\binom{n}{n}\right)\right) \\
&= \left(\frac{-i}{2}\right)^n \left((1+(-1)^n)(n-1) + 1 + (-1)^n\right) \\
&= \left(\frac{-i}{2}\right)^n (1+(-1)^n)\, n = \frac{1+(-1)^n}{2^n}\cos\frac{n\pi}{2}.
\end{aligned}$$

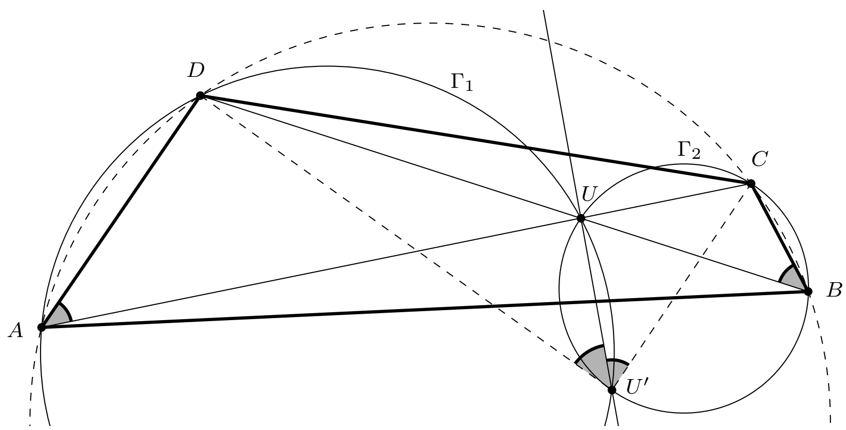
[*Ed.: If* n *is odd then* $\cos\frac{n\pi}{2} = 0 = 1+(-1)^n$*; if* $n = 4k$*, then* $\cos\frac{n\pi}{2} = 1 = (-i)^n$*; and if* $n = 4k+2$*, then* $\cos\frac{n\pi}{2} = -1 = (-i)^2 = (-i)^n$*.*]

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; CALAH PAULHUS *and IRINA STALLION, students, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. All these solutions are similar to, or virtually the same as the one featured above.*

3678. [2011 : 454, 456] *Proposed by Michel Bataille, Rouen, France.*

Let Γ_1, Γ_2 be two intersecting circles and U one of their common points. Show that there exists infinitely many pairs of lines passing through U and meeting Γ_1 and Γ_2 in four concyclic points. Give a construction of such pairs.

Solution by John G. Heuver, Grande Prairie, AB.



Suppose the problem is solved and let $ABCD$ be the cyclic quadrilateral and let U' be the second point of intersection of the two circles. Consider diagonal AC passing through U with A on Γ_1 and C on Γ_2. Note that

$$\angle UU'C = \angle UBC = \angle DBC = \angle DAC = \angle DAU = \angle DU'U.$$

It follows that $\angle UU'C = \angle DU'U$. Hence, we can construct point D as the point of intersection of Γ_1 with the image of the ray $U'C$ in reflection in $U'U$. Then B is the intersection of DU with Γ_2. This construction produces cyclic quadrilateral $ABCD$, once AC through U is determined. Since diagonal AC through U was chosen arbitrarily, it follows that there are infinitely many cyclic quadrilaterals satisfying the condition of the problem.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; ROY BARBARA and GEORGES MELKI, Fanar, Lebanon; OLIVER GEUPEL, Brühl, NRW, Germany; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; and the proposer.

3679. *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$(a^2b + c)(b^2c + a)(c^2a + b) \ \leq\ 4(ab + bc + ca - abc)\,.$$

Ed.: It seems the proposer's solution was flawed and we received only one other solution. Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, submitted a computer assisted solution. Perfetti's solution transformed the original problem into one of showing that

$$\sum_{k=0}^{7} P_k(x,y)c^k \le 0$$

where $x = a - c$, $y = b - c$ and each P_k is a homogeneous polynomial in x and y of degree $9 - k$. He then went on to show that each $P_k(x,y) \le 0$ for any x, y, k.

As we have not received any other "nice" solutions it is possible that one does not exist. We will leave the problem open in case any ingenious **Crux** reader has a flash of insight.

3680. [2011 : 454, 457] *Proposed by Michel Bataille, Rouen, France.*

In a system of axes (Ox, Oy, Oz), let $U(1,1,1)$, $S(a,b,c)$ and $H(h_a, h_b, h_c)$ where a, b, c are the sides of a triangle ABC and h_a, h_b, h_c are the corresponding altitudes. Given that the lines OU and SH intersect at M such that $|HM| = \frac{1}{3}|HS|$, find the angles of ΔABC.

Solution by the proposer.

The angles of ΔABC are $\frac{\pi}{6}, \frac{\pi}{6}$, and $\frac{2\pi}{3}$. To see this we write $\overrightarrow{HM} = t\overrightarrow{HS}$ (where $|t| = \frac{1}{3}$) and obtain

$$M\left(ta + (1-t)h_a,\ tb + (1-t)h_b,\ tc + (1-t)h_c\right).$$

Furthermore, because M is on the line OU, we have

$$ta + (1-t)h_a = tb + (1-t)h_b = tc + (1-t)h_c. \tag{1}$$

Eliminating t from (1) yields $(h_b - h_c)(b - a + h_a - h_b) = (h_a - h_b)(c - b + h_b - h_c)$, whence

$$(h_b - h_c)(b - a) = (h_a - h_b)(c - b). \tag{2}$$

Since the area of ΔABC is $F = \dfrac{ah_a}{2} = \dfrac{bh_b}{2} = \dfrac{ch_c}{2}$, equation (2) easily reduces to

$$\frac{(c-b)(b-a)}{bc} = \frac{(b-a)(c-b)}{ab},$$

from which it follows that $(b-a)(a-c)(c-b) = 0$ and, thus, ΔABC is isosceles. Suppose that the triangle has been labeled so that $b = c$. Note that a can equal neither b nor c (otherwise the lines OU and SH would coincide), so that $h_a - h_b \neq 0$ and $b - a + h_a - h_b \neq 0$ (because $b - a$ and $h_a - h_b$ are either both positive or both negative). Now (1) gives

$$t = \frac{1}{1 + \dfrac{b-a}{h_a - h_b}} = \frac{1}{1 + \dfrac{b-a}{\frac{2F}{a} - \frac{2F}{b}}} = \frac{1}{1 + \dfrac{ab}{2F}} = \frac{1}{1 + \frac{1}{\sin C}}.$$

Thus $t > 0$, so that $t = \frac{1}{3}$ and, therefore, $\sin C = \frac{1}{2}$. Since a base angle C of an isosceles triangle cannot be obtuse, we conclude that $C = B = \frac{\pi}{6}$ and $a = \frac{2\pi}{3}$, as claimed.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA.