RECURRING CRUX CONFIGURATIONS 8

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Heronian Triangles

We call a triangle *Heronian* if its sides and area are all integers; it is called *rational* when the sides and area are rational numbers. (Unfortunately, not even **Crux** has been consistent with this terminology; the reader should note that some sources require only rational sides and area for a triangle to be Heronian.) It is clear that any rational triangle is similar to a Heronian triangle—because the altitudes of a rational triangle are necessarily rational, it suffices to multiply each side and one of the altitudes by twice the common denominator. While the literature devoted to this topic is extensive, Heronian triangles have appeared in these pages in only four problems and one short note. The note [1982: 206] was "An Heronian Oddity" by Leon Bankoff. The author discussed the 13-14-15 Heronian triangle, which is formed by placing a 5-12-13 right triangle beside a 9-12-15 right triangle so that they share the leg of length 12. The common leg then serves as the altitude to the side of length $5+9$ of the 13-14-15 triangle, while extending the string of consecutive integers. Bankoff remarked further that the inradii of the 5-12-13, the 9-12-15, and the 13-14-15 triangles are also measured by consecutive integers: 2, 3, and 4, respectively. This observation may not be very deep, but it becomes important if you run out of other things to talk about while on a date.

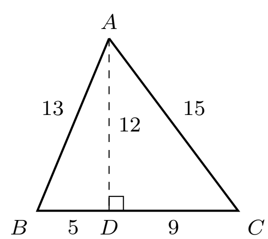

Figure 1: The juxtaposition of integer-sided right triangles ABD and ADC to form a Heronian triangle ABC.

It turns out that *all* Heronian triangles and *only* Heronian triangles are obtained either by the juxtaposition of two integer-sided right triangles as in the figure, or by the reduction of such a juxtaposition by a common factor. Remarkably, according to David Singmaster [1985: 222-223] this result was not completely proved until 1978 in [3]; see also [2]. That proof depends on the following theorem.

Theorem. Let k, a, b, c be positive integers. Then a, b, c are the sides of a Heronian triangle if and only if ka, kb, kc form a Heronian triangle.

As a consequence, a triangle is Heronian if and only if its sides can be represented either as

$$a = m(u^2 - v^2) + n(r^2 - s^2), \quad b = m(u^2 + v^2), \quad c = n(r^2 + s^2), \qquad (1)$$

for positive integers m, n, r, s, u, v satisfying $muv = nrs$, or as a reduction by a common factor of a triangle given by (1). The altitude to the side a of the triangle in (1) is the common product $muv = nrs$. Singmaster's comments were his contribution to the solution of the following problem, the problem that inspired Bankoff's musings on consecutive-integer triangles that we reproduced above.

Problem 290 [1977: 251; 1978: 142-147; 1985: 222-223] (proposed by R. Robinson Rowe). Find a 9-digit integer representing the area of a triangle of which the three sides are consecutive integers.

The solution by Clayton W. Dodge characterized those Heronian triangles whose sides are consecutive integers. Specifically, they are those triangles whose middle side and area are

$$a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n, \qquad \text{Area } = \frac{3}{4\sqrt{3}}\left((2+\sqrt{3})^{2n} - (2-\sqrt{3})^{2n}\right).$$

Examples can be calculated by hand using the recurrence relations

$$\begin{array}{llll} x_1 = 2, & x_2 = 7, & x_{n+2} = 4x_{n+1} - x_n, & n = 1, 2, \cdots ; \quad \text{and} \\ y_1 = 1, & y_2 = 4, & y_{n+2} = 4y_{n+1} - y_n, & n = 1, 2, \cdots . \end{array}$$

The middle side of the nth triangle will be $a_n = 2x_n$, and its area $A_n = 3x_n y_n$. The first four Heronian triangles whose sides are consecutive integers have middle side and area as follows:

n	a_n	A_n
1	4	6
2	14	84
3	52	1170
4	194	16296

For the record, the 9-digit area requested by Rowe is $A_8 = 613,283,664$. Dodge's solution was followed by two pages of comments by the editor, most of which summarized the information related to this problem found in the 11 pages of [1] that Dickson devoted to rational triangles.

Problem 1148 [1987: 108; 1988: 83; 1991: 302-303] (proposed by Stanley Rabinowitz). Find the triangle of smallest area that has integral sides and integral altitudes.

The smallest such triangle is the right triangle with sides $15, 20, 25$, altitudes $20, 15, 12$, and area 150. A nice proof was supplied in 1991 by Sam Maltby, who as a bonus proved that

- The smallest non-right triangles with integral sides and altitudes are the 25-25-30 (acute) and 25-25-40 (obtuse) triangles with areas of 300 each and altitudes 24,24,20 and 24,24,15, respectively.

- The triangle of smallest area having integral sides, area, circumradius, and inradius (described as an open problem in [1, p. 200]) is the 6-8-10 triangle with area 24, $R = 5$, and $r = 2$. If we add the restriction that the altitudes must also be integral, we get the 30-40-50 triangle.

Problem 2764 [2002: 397; 2003: 348-349] (proposed by Christopher J. Bradley). Find an integer-sided scalene triangle in which the lengths of the internal angle bisectors all have integer lengths.

I found four typographical errors in the last paragraph of the featured solution. The example there is a Heronian triangle whose sides are 10-digit numbers; perhaps with numbers so large the introduction of errors should come as no big surprise. A more appealing problem would have called for examples with rational sides and rational bisectors, allowing the reader to multiply by a common denominator to satisfy his own curiosity. That, in fact, describes problem E 418 in [4]. The solution by E.P. Starke included the comment that *if the sides and internal angle bisectors are rational, so also are the external angle bisectors, the altitudes, the area, R, r, and the three exradii.* His solution began with two rational solutions (x_1, y_1) and (x_2, y_2) of $x^2 + y^2 = 1$, with $x_1x_2 > y_1y_2$. Here x_1 plays the role of $\cos\frac{B}{2}$, x_2 of $\cos\frac{C}{2}$. Then any numbers a, b, c proportional to

$$(x_1x_2 - y_1y_2)(x_1y_2 + y_1x_2), \quad x_1y_1, \quad x_2y_2$$

are the sides of a triangle whose angle bisectors are

$$\frac{2bc(x_1y_2 + y_1x_2)}{b+c}, \quad \frac{2cax_1}{c+a}, \quad \frac{2abx_2}{a+b}.$$

The characterization in (1) corresponds to Starke's by replacing the integers m, u, v by the rational numbers x_1y_1, x_2, y_2, respectively, as well as n, r, s by x_2y_2, x_1, y_1. For example, the solutions $\left(\frac{3}{5}, \frac{4}{5}\right)$ and $\left(\frac{12}{13}, \frac{5}{13}\right)$ produce a scalene triangle with sides 84, 169, 125, and angle bisectors $\frac{975}{7}, \frac{12600}{209}, \frac{26208}{253}$. If you wish, you may multiply through by the common denominator $7 \cdot 11 \cdot 19 \cdot 23$ to get a triangle whose sides and bisectors are 7-digit integers.

The fourth **Crux** problem dealing with Heronian triangles, number 2765 [2002: 397; 2003: 349-351], involves triangles whose nine-point centre lies on side BC; it will be discussed in the ninth essay of our series.

References

[1] Leonard Eugene Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952, Volume II, pp. 191-201.

[2] David Singmaster, Some Corrections to Carlson's "Determination of Heronian Triangles", *The Fibonacci Quarterly*, **11**:2 (April 1973) 157-158.

[3] David Singmaster, Letter to the Editor, *Mathematical Spectrum*, **11** (1978/1979) 58-59.

[4] E.P. Stark, Solution to problem E 418, *American Mathematical Monthly*, **48**:1 (Jan. 1941) 67-68. (Problem proposed by W.E. Buker, 1940, p. 240.)