

THE OLYMPIAD CORNER

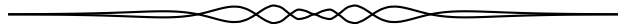
No. 306

Nicolae Strungaru

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 février 2014**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.



OC96. Soient $a, b > 1$ deux entiers relativement premiers. On pose $x_1 = a$, $x_2 = b$ et

$$x_n = \frac{x_{n-1}^2 + x_{n-2}^2}{x_{n-1} + x_{n-2}}$$

pour $n \geq 3$. Démontrer que x_n est entier pour aucune valeur $n \geq 3$.

OC97. Soit A un ensemble de 225 éléments. Supposons qu'il existe onze sous-ensembles A_1, \dots, A_{11} de A tels que $|A_i| = 45$ pour $1 \leq i \leq 11$ et $|A_i \cap A_j| = 9$ pour $1 \leq i < j \leq 11$. Démontrer que $|A_1 \cup A_2 \cup \dots \cup A_{11}| \geq 165$, et donner un exemple où l'égalité tient.

OC98. Soit ABC un triangle tel que $\angle BAC = 60^\circ$. Soient B_1 et C_1 les pieds des bissectrices partant de B et C . Soit A_1 symétrique à A par rapport à la ligne B_1C_1 . Démontrer que A_1, B et C sont colinéaires.

OC99. Soit \mathbb{Q}^+ l'ensemble des nombres rationnels positifs. Déterminer toutes les fonctions $f : \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$ telles que pour tout $x \in \mathbb{Q}^+$ les égalités

$$f\left(\frac{x}{x+1}\right) = \frac{f(x)}{x+1}$$

et

$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^3}$$

tiennent.

OC100. Soit a_n la suite définie par $a_0 = 1$, $a_1 = -1$, et

$$a_n = 6a_{n-1} + 5a_{n-2}$$

pour $n \geq 2$. Démontrer que $a_{2012} - 2010$ est divisible par 2011.

OC96. Let $a, b > 1$ be two relatively prime integers. We define $x_1 = a$, $x_2 = b$ and

$$x_n = \frac{x_{n-1}^2 + x_{n-2}^2}{x_{n-1} + x_{n-2}}$$

for all $n \geq 3$. Prove that x_n is not an integer for all $n \geq 3$.

OC97. Let A be a set with 225 elements. Suppose that there are eleven subsets A_1, \dots, A_{11} of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \dots \cup A_{11}| \geq 165$, and give an example for which equality holds.

OC98. Let ABC be a triangle with $\angle BAC = 60^\circ$. Let B_1 and C_1 be the feet of the bisectors from B and C . Let A_1 be the symmetrical of A with respect to the line B_1C_1 . Prove that A_1, B and C are collinear.

OC99. Let \mathbb{Q}^+ denote the set of positive rational numbers. Determine all functions $f: \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$ so that, for all $x \in \mathbb{Q}^+$ we have

$$f\left(\frac{x}{x+1}\right) = \frac{f(x)}{x+1}$$

and

$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^3}.$$

OC100. Let a_n be the sequence defined by $a_0 = 1$, $a_1 = -1$, and

$$a_n = 6a_{n-1} + 5a_{n-2}$$

for all $n \geq 2$. Prove that $a_{2012} - 2010$ is divisible by 2011.

OLYMPIAD SOLUTIONS

OC36. The obtuse-angled triangle ABC has sides of length a , b , and c opposite the angles $\angle A$, $\angle B$ and $\angle C$ respectively. Prove that

$$a^3 \cos A + b^3 \cos B + c^3 \cos C < abc.$$

(Originally question #6 from the 2008/9 British Mathematical Olympiad, Round 1.)

Similar solutions by Michel Bataille, Rouen, France and Titu Zvonaru, Comănești, Romania. No other solution was received.

By the law of cosines we have

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}; \cos(B) = \frac{a^2 + c^2 - b^2}{2ac}; \cos(C) = \frac{a^2 + b^2 - c^2}{2ab}.$$

Then the inequality reduces to

$$a^4b^2 + a^2b^4 + b^4c^2 + b^2c^4 + c^4a^2 + c^2a^4 - (a^6 + b^6 + c^6 + 2a^2b^2c^2) < 0$$

or equivalently

$$(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)(a^2 + b^2 - c^2) < 0$$

which is true since ABC is obtuse angled, and thus exactly one of $\cos(A)$, $\cos(B)$, $\cos(C)$ is negative.

Zvonaru observed that using the same argument we can prove that if ABC is acute angled then

$$a^3 \cos(A) + b^3 \cos(B) + c^3 \cos(C) > abc$$

and if ABC is right triangle then

$$a^3 \cos(A) + b^3 \cos(B) + c^3 \cos(C) = abc.$$

OC37. Find all integers n such that we can colour all the edges and diagonals of a convex n -gon by n given colours satisfying the following conditions:

- (i) Every one of the edges or diagonals is coloured by only one colour;
- (ii) For any three distinct colours, there exists a triangle whose vertices are vertices of the n -gon and the three edges are coloured by the three colours, respectively.

(Originally question #5 from the 2009 Chinese Mathematical Olympiad.)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We prove that an integer $n \geq 3$ has the desired property if and only if n is odd.

Suppose that n has the required property. We show that n is odd. There are $\binom{n}{3}$ choices of three distinct colours out of n colours and also $\binom{n}{3}$ triangles out of n vertices. Hence, any two distinct such triangles have distinct sets of edge colours.

As there are exactly $\binom{n-1}{2}$ choices of three colours containing a fixed given color, each colour occurs in exactly $\binom{n-1}{2}$ triangles. On the other hand, every line segment belongs to $n-2$ triangles. Therefore, the number of segments of each colour is $\frac{1}{n-2} \binom{n-1}{2} = \frac{n-1}{2}$. We conclude that n is odd.

Next suppose that $n \geq 3$ is odd. We prove that n has the desired property. Suppose that there are n colours C_0, C_1, \dots, C_{n-1} and n vertices P_0, P_1, \dots, P_{n-1} . Consider indices of colours and indices of vertices modulo n . Let us colour the edge with end points P_i and P_j with colour C_{i+j} . Let C_i, C_j , and C_k be any three distinct colours. It is now straightforward to verify that the triangle $P_{(i+j-k)/2}P_{(i+k-j)/2}P_{(j+k-i)/2}$ has edges with colours C_i, C_j , and C_k . (Here the calculation of indices is within the additive ring \mathbb{Z}_n . As n is odd, 2 is invertible in \mathbb{Z}_n .) This completes the proof.

OC38. Let a, b, c be positive real numbers such that $ab + bc + ca = \frac{1}{3}$. Prove the inequality:

$$\frac{a}{a^2 - bc + 1} + \frac{b}{b^2 - ca + 1} + \frac{c}{c^2 - ab + 1} \geq \frac{1}{a + b + c}.$$

(Originally question #4 from the 16th Macedonian Mathematical Olympiad.)

Solved by John Asmanis, Chalkida, Greece; Michel Bataille, Rouen, France; Marian Dincă, Bucharest, Romania; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Henry Ricardo, Tappan, NY, USA and Titu Zvonaru, Comănești, Romania. We give the solution of Bataille.

First, we observe that $a^2 - bc + 1 = a^2 + ab + ac - (ab + ac + bc) + 1 = a(a + b + c) + \frac{2}{3}$.

We denote by $s := a + b + c$. Then, the inequality becomes equivalent to

$$\frac{1}{s^2} \leq \frac{a}{s} \cdot \frac{1}{as + \frac{2}{3}} + \frac{b}{s} \cdot \frac{1}{bs + \frac{2}{3}} + \frac{c}{s} \cdot \frac{1}{cs + \frac{2}{3}}. \tag{1}$$

Since $f(x) = \frac{1}{x}$ is convex on $(0, \infty)$ and $\frac{a}{s} + \frac{b}{s} + \frac{c}{s} = 1$ by the Jensen inequality we get

$$\begin{aligned} \frac{a}{s} \cdot \frac{1}{as + \frac{2}{3}} + \frac{b}{s} \cdot \frac{1}{bs + \frac{2}{3}} + \frac{c}{s} \cdot \frac{1}{cs + \frac{2}{3}} &\geq \frac{1}{\frac{a}{s}(as + \frac{2}{3}) + \frac{b}{s}(bs + \frac{2}{3}) + \frac{c}{s}(cs + \frac{2}{3})} \\ &= \frac{1}{a^2 + b^2 + c^2 + \frac{2}{3}} \\ &= \frac{1}{a^2 + b^2 + c^2 + 2(ab + ac + bc)} = \frac{1}{s^2}, \end{aligned}$$

which proves (1).

OC39. Given a positive integer n , let $b(n)$ denote the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n . For example $b(13) = 6$ because $13 = 1101_2$, which contains as consecutive blocks the binary representations of $13 = 1101_2$, $6 = 110_2$, $5 = 101_2$, $3 = 11_2$, $2 = 10_2$ and $1 = 1_2$.

Show that if $n \leq 2500$, then $b(n) \leq 39$, and determine the values of n for which equality holds.

(Originally question #4 from the 2008/9 British Mathematical Olympiad, Round 2.)

Solved by Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Comănești, Romania. We give a combination of their similar solutions.

For any positive integers ℓ and m and any bit string $x \in \{0, 1\}^m$ (i.e., a word of length m over the alphabet $\{0, 1\}$), let $B(x, \ell)$ denote the number of distinct bit strings of length ℓ and with leading bit 1 that occur as consecutive blocks in x . Let $B(x) = \sum_{\ell=1}^{\infty} B(x, \ell)$. Moreover, let $\beta(n)$ denote the binary (bit string) representation of the integer n . The identity $B(\beta(n)) = b(n)$ clearly holds.

Claim 1: If $2^{11} \leq n \leq 2050$ then $b(n) \leq 37$.

Indeed, $x = \beta(n)$ has the form $100x_9$ with $x_9 \in \{0, 1\}^9$. By counting the possible bit strings, we obtain $B(x, 1) = 1$, $B(x, 2) \leq 2$, $B(x, 3) \leq 4$. We have $B(x, 4) \leq 7$, because there are not more than 6 blocks inside x_9 and one additional block starting at the leftmost bit of x . Similarly, we obtain $B(x, 5) \leq 6$, $B(x, 6) \leq 5$, $B(x, 7) \leq 4$, $B(x, 8) \leq 3$, $B(x, 9) \leq 2$, and $B(x, 10) = B(x, 11) = B(x, 12) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 37,$$

which completes the proof of Claim 1.

Now we assume that $n < 2048$.

Claim 2: If $n \leq 2^{10}$ then $b(n) \leq 36$.

By a similar counting to the one in Claim 1, we obtain $B(x, 1) = 1$, $B(x, 2) \leq 2$, $B(x, 3) \leq 4$, $B(x, 4) \leq 8$.

As a block of length 5 in $x = x_1x_2\dots x_{10}$ can only start at x_1, x_2, x_3, x_4, x_5 or x_6 we get $B(x, 5) \leq 6$ and similarly $B(x, 6) \leq 5$, $B(x, 7) \leq 4$, $B(x, 8) \leq 3$, $B(x, 9) \leq 2$, and $B(x, 10) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 36,$$

which completes the proof of Claim 2.

Claim 3: If $n \leq 2^{11}$ and $x = \beta(n)$, it holds $B(x, 1) \leq 1$, $B(x, 2) \leq 2$, $B(x, 3) \leq 4$, $B(x, 4) \leq 8$, $B(x, 5) \leq 7$, $B(x, 6) \leq 6$, $B(x, 7) \leq 5$, $B(x, 8) \leq 4$, $B(x, 9) \leq 3$, $B(x, 10) \leq 2$, and $B(x, 11) \leq 1$.

The first four relations follow immediately from the observation that, by counting all the binary strings of length k starting with 1 we have

$$B(x, k) \leq 2^{k-1}.$$

For the others it suffices to observe that there are exactly $11 - k + 1$ possible starting positions for any block of length k in x . Thus

$$B(X, k) \leq 12 - k.$$

This proves the claim.

Now we assume that $1024 \leq n < 2048$. Then $x := \beta(n) = 1x_2x_3\dots x_{11}$.

We split the problem in four cases:

Case 1: $x_2 = 0$. Then, $x := \beta(n) = 10x_3\dots x_{11}$, and exactly as in Claim 1 we get $B(x, 1) = 1$, $B(x, 2) \leq 2$, $B(x, 3) \leq 4$, $B(x, 4) \leq 7$, $B(x, 5) \leq 6$, $B(x, 6) \leq 5$, $B(x, 7) \leq 4$, $B(x, 8) \leq 3$, $B(x, 9) \leq 2$, and $B(x, 10) = B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 36.$$

Case 2: $x_2 = 1, x_3 = 0$. Then $x := \beta(n) = 110x_4\dots x_{11}$, and exactly as in Claim 1 we get $B(x, 1) = 1$, $B(x, 2) \leq 2$, $B(x, 3) \leq 4$, $B(x, 4) \leq 7$, $B(x, 5) \leq 6$, $B(x, 6) \leq 5$, $B(x, 7) \leq 4$, $B(x, 8) \leq 3$, $B(x, 9) \leq 2$, $B(x, 10) \leq 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 37.$$

Case 3: $x_2 = 1, x_3 = 1, x_4 = 0$. Then $x := \beta(n) = 1110x_5\dots x_{11}$, and exactly as in Claim 1 we get $B(x, 1) = 1$, $B(x, 2) \leq 2$, $B(x, 3) \leq 4$, $B(x, 4) \leq 7$, $B(x, 5) \leq 6$, $B(x, 6) \leq 5$, $B(x, 7) \leq 4$, $B(x, 8) \leq 3$, $B(x, 9) \leq 3$, $B(x, 10) \leq 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 38.$$

Case 4: $x_2 = 1, x_3 = 1, x_4 = 1$. We will also split this case in 7 subcases:

SubCase 4a: $x_5 = x_6 = 0$. Then $x := \beta(n) = 111100x_7\dots x_{11}$.

Then, $B(x, 5) \leq 5$, $B(x, 6) \leq 5$, $B(x, 7) \leq 4$, $B(x, 8) \leq 4$, and hence from Claim 3 we get

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 38.$$

SubCase 4b: $x_5 = 0, x_6 = 1, x_7 = 0$. Then $x := \beta(n) = 1111010x_8x_9x_{10}x_{11}$.

$B(x, 1) = 1$, $B(x, 2) \leq 2$, $B(x, 3) \leq 4$, $B(x, 4) \leq 6$, $B(x, 5) \leq 5$, $B(x, 6) \leq 5$, $B(x, 7) \leq 4$, $B(x, 8) \leq 4$, $B(x, 9) \leq 3$, $B(x, 10) \leq 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 37.$$

SubCase 4c: $x_5 = 0, x_6 = 1, x_7 = 1$. Then $x := \beta(n) = 1111011x_8x_9x_{10}x_{11}$.

$B(x, 1) = 1, B(x, 2) \leq 2, B(x, 3) \leq 4, B(x, 4) \leq 7, B(x, 5) \leq 6, B(x, 6) \leq 5, B(x, 7) \leq 4, B(x, 8) \leq 4, B(x, 9) \leq 3, B(x, 10) \leq 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 39.$$

To get equality we must have $B(x, 1) = 1, B(x, 2) = 2, B(x, 3) = 4, B(x, 4) = 7, B(x, 5) = 6, B(x, 6) = 5, B(x, 7) = 4, B(x, 8) = 4, B(x, 9) = 3, B(x, 10) = 2$, and $B(x, 11) = 1$.

If $x_8 = 0$, we get $B(x, 4) \leq 6$, thus $b(n) \leq 38$.

Thus we can only get equality if $x_8 = 1$. Then $x = 11110111x_9x_{10}x_{11}$. Since $B(x, 3) = 4$ we must have all 4 blocks 100, 101, 110, 111 in the binary representation of n . Hence either $x_9 = x_{10} = 0$ or $x_{10} = x_{11} = 0$, which shows that equality can only occur for

$$x \in \{11110111000, 11110111001, 11110111100\} = X.$$

As 11110111100 doesn't contain 1000 and 1001, $B(11110111100, 4) \neq 7$, and hence $B(11110111100) \neq 39$.

As 11110111001 doesn't contain 1010 and 1000, $B(11110111100, 4) \neq 7$, and hence $B(11110111100) \neq 39$.

As 11110111100 doesn't contain 1010 and 1000, $B(11110111100, 4) \neq 7$, and hence $B(11110111100) \neq 39$.

Thus, there is no solution in this case.

SubCase 4d: $x_5 = 1, x_6 = 0, x_7 = 0$. Then $x := \beta(n) = 1111100x_8x_9x_{10}x_{11}$.

$B(x, 1) = 1, B(x, 2) \leq 2, B(x, 3) \leq 4, B(x, 4) \leq 5, B(x, 5) \leq 5, B(x, 6) \leq 5, B(x, 7) \leq 4, B(x, 8) \leq 4, B(x, 9) \leq 3, B(x, 10) \leq 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 37.$$

SubCase 4e: $x_5 = 1, x_6 = 0, x_7 = 1$. Then $x := \beta(n) = 1111101x_8x_9x_{10}x_{11}$.

$B(x, 1) = 1, B(x, 2) \leq 2, B(x, 3) \leq 4, B(x, 4) \leq 6, B(x, 5) \leq 6, B(x, 6) \leq 5, B(x, 7) \leq 5, B(x, 8) \leq 4, B(x, 9) \leq 3, B(x, 10) \leq 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 39.$$

To get equality we must have $B(x, 1) = 1, B(x, 2) = 2, B(x, 3) = 4, B(x, 4) = 6, B(x, 5) = 6, B(x, 6) = 5, B(x, 7) = 5, B(x, 8) = 4, B(x, 9) = 3, B(x, 10) = 2$, and $B(x, 11) = 1$.

If $x_8 = 0$, then $x = 11111010x_9x_{10}x_{11}$ we get $B(x, 4) \leq 5$, thus $b(n) \leq 38$.

Thus we can only get equality if $x_8 = 1$. Then $x = 11111011x_9x_{10}x_{11}$. Since $B(x, 3) = 4$ we must have all 4 blocks 100, 101, 110, 111 in the binary representation

of n . Hence either $x_9 = x_{10} = 0$ or $x_{10} = x_{11} = 0$, which shows that equality can only occur for

$$x \in \{11111011000, 11111011001, 11111011100\}.$$

It is easy to check that both $x = 11111011000$ and $x = 11111011001$ work. Thus, in this case, we get equality for $n = 2008$ and $n = 2009$.

SubCase 4f: $x_5 = 1, x_6 = 1, x_7 = 0$. Then $x := \beta(n) = 1111110x_8x_9x_{10}x_{11}$.

$B(x, 1) = 1, B(x, 2) \leq 2, B(x, 3) \leq 4, B(x, 4) \leq 5, B(x, 5) \leq 5, B(x, 6) \leq 6, B(x, 7) \leq 5, B(x, 8) \leq 4, B(x, 9) \leq 3, B(x, 10) \leq 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 38.$$

SubCase 4g: $x_5 = 1, x_6 = 1, x_7 = 1$. Then $x := \beta(n) = 1111111x_8x_9x_{10}x_{11}$.

$B(x, 1) = 1, B(x, 2) \leq 2, B(x, 3) \leq 4, B(x, 4) \leq 5, B(x, 5) \leq 5, B(x, 6) \leq 6, B(x, 7) \leq 5, B(x, 8) \leq 4, B(x, 9) \leq 3, B(x, 10) \leq 2$, and $B(x, 11) = 1$. Consequently,

$$b(n) = B(x) = \sum_{\ell=1}^{12} B(x, \ell) \leq 38.$$

This proves that for all $n \leq 2500$ we have $b(n) \leq 39$ with equality if and only if

$$n \in \{2008, 2009\}.$$

OC40. Let M and N be the intersection of two circles, Γ_1 and Γ_2 . Let AB be the line tangent to both circles closer to M , say $A \in \Gamma_1$ and $B \in \Gamma_2$. Let C be the point symmetrical to A with respect to M , and D the point symmetrical to B with respect to M . Let E and F be the intersections of the circle circumscribed around DCM and the circles Γ_1 and Γ_2 , respectively.

Show that the circles circumscribed around the triangles MEF and NEF have radii of the same length.

(Originally question #5 from the 2009 Italian Team Selection Test.)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Comănești, Romania. We give the solution of Zvonaru.

Let P be the intersection point of AB and MN . Using the power of a point with respect to a circle we get

$$PA^2 = PM \cdot PN = PB^2.$$

Thus P is the midpoint of AB . As M is the midpoint of BD we get

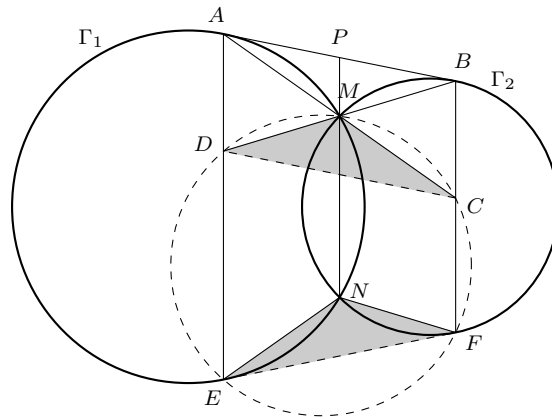
$$AD \parallel PM \parallel BC.$$

Combining the fact that M, D, E, F, C are on the same circle with the fact that $ABCD$ is a parallelogram and AB tangent to Γ_1 , we get

$$\angle DEM = \angle DCM = \angle MAB = \angle AEM,$$

$$\angle CFM = \angle MDC = \angle MBA = \angle BFM.$$

Hence A, E, M are collinear and B, C, F are also collinear.



It results that the trapezoids $DEFC$, $AENM$, $BFNM$ are cyclic, hence isosceles. It follows that

$$DC = EF; EN = AM = MC; NF = BM = MD.$$

Thus the triangles NEF and MCD are congruent. Since the triangles MEF and MCD are inscribed in the same circle, the statement of the problem follows.

