PROBLEM OF THE MONTH

No. 2

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This column is dedicated to the memory of former CRUX with MAYHEM Editor-in-Chief Jim Totten. Jim shared his love of mathematics with his students, with readers of CRUX with MAYHEM, and, through his work on mathematics contests and outreach programs, with many others. The “Problem of the Month” features a problem and solution that we know Jim would have liked.

As this column is dedicated to the memory of Jim Totten, what can be more fitting than using one of his own problems? When Jim was a graduate student, he was assigned an office near Ross Honsberger’s office. As a result, professor Honsberger often shared his “gems” with Jim. When Jim began teaching he wanted to share some great problems with his students, which he did through a “Problem of the Week”. Jim continued this practice throughout his career. This issue’s featured problem comes from the CMS ATOM series (A Taste Of Mathematics), Volume VII, “Problems of the Week” by Jim Totten.

Given a square $ABCD$ with $E$ the mid-point of side $CD$. Join $A$ to $E$ and drop a perpendicular from $B$ to $AE$ at $F$. Prove $CF = CD$.

What makes a problem beautiful? As with anything, beauty is in the eye of the beholder. I tend to be drawn to problems that submit to many different solutions, which is the case with this problem.

Solution #1: Since $ABCD$ is a square, then $\angle FAB$ and $\angle DAE$ are complementary, hence $\angle FAB = \angle AED$ and thus right angled triangles $\Delta FAB$ and $\Delta DEA$ are similar. So, if we let the side length of the square be $s$, then $DA = s$, $ED = \frac{s}{2}$ and $AE = \frac{s\sqrt{5}}{2}$ by the Pythagorean theorem. Thus, by similarity we have

$$\frac{FA}{DE} = \frac{AB}{EA} \Rightarrow \frac{2FA}{s} = \frac{2s}{\sqrt{5}s} \Rightarrow FA = \frac{s}{\sqrt{5}}.$$

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Drop a perpendicular from $F$ to $CD$ at $G$. Now the homothetic triangles $GFE$ and $DAE$ are similar and $FE = \frac{\sqrt{5}s}{2} - \frac{s}{\sqrt{5}}$, hence

$$\frac{GF}{DA} = \frac{FE}{AE} \Rightarrow \frac{GF}{s} = \frac{\frac{\sqrt{5}s}{2} - \frac{s}{\sqrt{5}}}{\frac{\sqrt{5}s}{2}} \Rightarrow GF = \frac{3s}{5}$$

and similarly, $EG = \frac{3s}{10}$.

Finally, we can apply the Pythagorean theorem to $\triangle CGF$, with $CG = \frac{s}{2} + \frac{3s}{10} = \frac{4s}{5}$ to get

$$CF^2 = \left(\frac{3s}{5}\right)^2 + \left(\frac{4s}{5}\right)^2$$

$$= \left(\frac{9}{25} + \frac{16}{25}\right)s^2$$

$$= s^2$$

hence $CF = s = CD$ completing the proof.

**Solution #2:** Since $\angle BFE = \angle ECB = 90^\circ$, quadrilateral $BFEC$ is cyclic. As in solution #1 we can determine $BC = s$, $CE = \frac{s}{2}$, $EF = \frac{\sqrt{5}s}{2} - \frac{s}{\sqrt{5}}$, $FB = \frac{2s}{\sqrt{5}}$ and $BE = \frac{\sqrt{5}s}{2}$. Thus, by Ptolemy’s theorem we have

$$CF \cdot BE = BC \cdot EF + CE \cdot FB$$

$$CF \left(\frac{\sqrt{5}s}{2}\right) = s \left(\frac{\sqrt{5}s}{2} - \frac{s}{\sqrt{5}}\right) + \left(\frac{s}{2}\right) \left(\frac{2s}{\sqrt{5}}\right)$$

$$CF = s - \frac{2}{5}s + \frac{2}{5}s,$$

hence $CF = s = CD$.

**Solution #3:** Continuing with the setup to the last solution, note that right-triangles $\triangle ADE$ and $\triangle BCE$ are congruent, and both similar to $\triangle BFA$. Thus,

$$\angle AED = \angle BEC = \angle BAF.$$ 

Since $\angle ABC$ is a right angle, angle $\angle ABF$ and $\angle FBC$ are complements, hence

$$\angle AED = \angle BEC = \angle BAF.$$ 

Thus as $\angle BEC = \angle FBC$ are equal inscribed angles, then the chords they subtend are equal, that is $CF = BC$. Thus, since $ABCD$ is a square $CD = BC = CF$ and we are done.
**Solution #4**: A reflection in the point $E$ takes $\triangle AED$ to $\triangle HEC$, where $H$ is the point where the extension of $AE$ meets the line $BC$.

Now $\triangle BFH$ is a right angled triangle so we can inscribe it in a circle with $BH$ as a diameter. Hence, the radius of the circle is $s$ and $CF = BC = CH = s$, thus $CF = CD$.

**Solution #5**: Let $J$ be the midpoint of $AB$ and let $CJ$ meet $BF$ at $K$. Then $CJ$ and $EA$ are parallel and thus $BF$ and $CJ$ are perpendicular and

\[
\frac{BK}{KF} = \frac{BJ}{JA} = 1,
\]

whence $BK = KF$. Now we have enough information to see that triangles $\triangle CKF$ and $\triangle CKB$ are congruent yielding $CF = BC$.

**Solution #6**: If we continue with the idea of the last solution, we see that if we start with a square $ABCD$ and connect $A$ to the midpoint of $CD$, $B$ to the midpoint of $AD$, $C$ to the midpoint of $AB$ and $D$ to the midpoint of $BC$, then pairs of these segments intersect at the corner of a smaller square. It doesn’t take much to see that we can create a “proof without words” creating a lattice of the little squares as in the diagrams below.

The obvious moral of the story is that when you have “solved” a problem, it is often worthwhile to take a second look. In some cases, like this month’s problem, a variety of solutions might be possible, some being more elegant or insightful or even more beautiful than others. Then again, beauty is in the eye of the beholder.