THE OLYMPIAD CORNER

No. 305

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The solutions to the problems are due to the editor by 1 January 2014.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions’ section, the problem will be stated in the language of the primary featured solution.

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OC91. Prove that no integer consisting of one 2, one 1 and the rest of digits 0 can be written either as the sum of two perfect squares or the sum of two perfect cubes.

OC92. Let $ABCD$ be a convex quadrilateral. Let $P$ be the intersection of external bisectors of $\angle DAC$ and $\angle DBC$. Prove that $\angle APD = \angle BPC$ if and only if $AD + AC = BC + BD$.

OC93. For every positive integer $n$, determine the maximum number of edges a simple graph with $n$ vertices can have if it contains no cycles of even length.

OC94. Let $x_1, x_2, \ldots, x_{25}$ be real numbers such that for all $1 \leq i \leq 25$ we have $0 \leq x_i \leq i$. Find the maximum value of

$$x_1^3 + x_2^3 + \cdots + x_{25}^3 - (x_1 x_2 x_3 + x_2 x_3 x_4 + \cdots + x_{25} x_1 x_2).$$

OC95. Can we find three relatively prime integers $a, b, c$ so that the square of each number is divisible by the sum of the other two?
OC94. Soient $x_1, x_2, \ldots, x_{25}$ des nombres réels tels que pour tout $1 \leq i \leq 25$, on a $0 \leq x_i \leq i$. Déterminer la valeur maximale de

$$x_1^3 + x_2^3 + \cdots + x_{25}^3 - (x_1 x_2 x_3 + x_2 x_3 x_4 + \cdots + x_{25} x_1 x_2).$$

OC95. Est-il possible de construire trois entiers relativement premiers, $a, b$, et $c$ tels que le carré de chacun d’entre eux est divisible par la somme des deux autres ?

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**OLYMPIAD SOLUTIONS**

OC31. Find all pairs $(p, q)$ of prime numbers such that $pq \mid (5^p + 5^q)$.

*(Originally question #2 from the 2009 Chinese Mathematical Olympiad.)*

*Solved by Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the similar solutions of Manes and Zelator.*

Note that if $(p, q)$ is a solution so is $(q, p)$ by symmetry. We show that the solutions are $(2, 3), (2, 5), (5, 5), (5, 313)$ and the pairs with swapped components. Because of this, we will seek only the solutions with $p \leq q$.

We break the problem in few cases:

*Case 1: $p = 2$. It is easy to see that $q = 2$ is not a solution in this case. Thus $q$ must be an odd prime in this case. Then $2q \mid (25 + 5^q)$ and hence

$$25 + 5^q \equiv 0 \pmod{q}. $$

By Fermat’s little theorem $5^q \equiv 5 \pmod{q}$. Thus

$$0 \equiv 25 + 5^q \equiv 25 + 5 \equiv 30 \pmod{q},$$

and hence $q \mid 30$. As $q$ is an odd prime, we get $q \in \{3, 5\}$. It is straightforward to check that these are solutions, thus in this case we get $(2, 3)$ and $(2, 5)$ as solutions.*

*Case 2: $p = q$ is an odd prime. Then $p^2 \mid 2 \cdot 5^p \Rightarrow p \mid 2 \cdot 5^p$. Since $p$ is an odd prime, we get $p = 5$, and it is easy to see that $p = q = 5$ is a solution. In this case we get the solution $(5, 5)$.*

*Case 3: $p < q$ are odd primes. We split this case in three subcases.*

*Subcase 3a: $p = 5$. Then

$$5^5 + 5^q \equiv 0 \pmod{q}.$$*

Since $\gcd(5, q) = 1$ we can cancel $5$ modulo $q$ and thus

$$5^4 + 5^{q-1} \equiv 0 \pmod{q}.$$
By Fermat’s little theorem we also get
\[ 0 \equiv 5^4 + 5^{q-1} \equiv 5^4 + 1 \equiv 626 \pmod{q} . \]

Thus \( q \mid 626 = 2 \cdot 313 \). As \( q \) is an odd prime, the only possible solution is \( q = 313 \).
We check now that this is indeed a solution.
By Fermat’s little theorem we have
\[ 5^4 + 5^{312} \equiv 5^4 + 1 \equiv 0 \pmod{313} , \]
and hence
\[ 5^5 + 5^{313} \equiv 0 \pmod{5 \cdot 313} . \]
Thus \( (5, 313) \) is the only solution in this subcase.

Subcase 3b: \( q = 5 \). As \( p < 5 \) is an odd prime we get that \( p = 3 \), and it is easy to check that \( (3, 5) \) is not a solution. Thus there is no solution in this subcase.

Subcase 3c: \( p \neq 5 \) and \( q \neq 5 \). We show there is no solution in this subcase. Assume by contradiction that \( (p, q) \) is a solution where \( p \neq q, p \neq 5, q \neq 5 \), with \( p \) and \( q \) odd.

As \( pq \mid 5^p + 5^q \) we get \( p \mid 5^p + 5^q \) and \( q \mid 5^p + 5^q \). Therefore, by Fermat’s little theorem we get
\[ 5^p + 5^q \equiv 0 \pmod{p} , \]
\[ 5^p + 5^q \equiv 0 \pmod{q} , \]
hence
\[ 5^{q-1} \equiv -1 \pmod{p} , \]
and
\[ 5^{p-1} \equiv -1 \pmod{q} . \]
Let \( e, f \) be the orders of 5 modulo \( p \) respectively \( q \). Then
\[ e \mid 2(q-1) ; e \nmid (q-1) , \]
\[ f \mid 2(p-1) ; f \nmid (p-1) . \]
By Fermat’s little theorem we also have
\[ e \mid (p-1) ; f \mid (q-1) . \]
Let \( a, b \) be the powers of 2 in \( p - 1 \) respectively \( q - 1 \). As \( e \mid 2(q-1) \) but \( e \nmid (q-1) \) we get \( 2^{b+1} \mid e \mid p - 1 \). Hence \( b + 1 \leq a \). Similarly, as \( f \mid 2(p-1) \) but \( f \nmid (p-1) \) we get \( 2^{a+1} \mid f \mid q - 1 \). Hence \( a + 1 \leq b \).

Thus
\[ b + 2 \leq a + 1 \leq b , \]
a contradiction.
Since we reached a contradiction, our assumption is wrong, and hence there is no solution in this subcase.
Thus all the solutions are
\[ \{(2, 3), (3, 2), (2, 5), (5, 2), (5, 5), (5, 313), (313, 5)\} . \]

**OC32.** Let \( \triangle ABC \) be an acute-angled triangle with \( \angle B = \angle C \). Let the circum-centre be \( O \) and the orthocentre be \( H \). Prove that the centre of the circle \( BOH \) lies on the line \( AB \).

*(Originally question #2 from the 2008/9 British Mathematical Olympiad, Round 2.)*

*Solved by Michel Bataille, Rouen, France; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Oliver Geupel, Brühl, NRW, Germany; Mihai-Ioan Stočnescu, Bischwiller, France; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA and Titu Zvonaru, Comănești, Romania. We give the solution of Geupel.*

Since \( AC \) is perpendicular to the altitude \( BH \), we have
\[ \angle ABH = 90^\circ - \angle A . \]

Let \( M \) be the centre of the circle \( BOH \). Then
\[ \angle MBH = 90^\circ - \frac{1}{2} \angle BMH = 90^\circ - \angle BOH . \]

Since \( \angle B = \angle C \), it holds
\[ \angle BOH = \frac{1}{2} \angle BOC = \angle A . \]

We deduce that
\[ \angle ABH = \angle MBH , \]
that is, the points \( A, B, \) and \( M \) are collinear.

**OC33.** Let \( n \) and \( k \) be integers such that \( n \geq k \geq 1 \). There are \( n \) light bulbs placed in a circle. They are all turned off. Each turn, you can change the state of any set of \( k \) consecutive light bulbs.

How many of the \( 2^n \) possible combinations can be reached.
(a) if \( k \) is an odd prime?

(b) if \( k \) is an odd integer?

(c) if \( k \) is an even integer?

(Originally question \#1 from 2009 Italian Team Selection Test.)

No solution to this problem was received.

**OC34.** Let \( m, n \) be integers with \( 4 < m < n \), and \( A_1A_2 \cdots A_{2n+1} \) be a regular \( 2n+1 \)-gon. Let \( P = \{ A_1, A_2, \ldots, A_{2n+1} \} \). Find the number of convex \( m \)-gons with exactly two acute internal angles whose vertices are all in \( P \).

(Originally question \#3 from the 2009 Chinese Mathematical Olympiad.)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We claim that the required number is \( (2n+1) \left[ \binom{n}{m-1} + \binom{n+1}{m-1} \right] \).

If \( \angle A_iA_jA_k \) is acute then the arc \( A_iA_k \) of the circumcircle that contains the point \( A_j \) is more than a half of the circle. Hence, if a convex \( m \)-gon contains two acute angles, then their vertices are neighbouring vertices of the \( m \)-gon.

Let \( \angle ABC \) and \( \angle BCD \) be the two acute angles. Assume for the moment that \( A = A_1 \). Also suppose that the smaller arc \( AD \) contains \( k \) points from \( P \) in its interior. There are \( \binom{k}{m-4} \) choices for \( m-4 \) points out of these \( k \) points.

The longer arc \( AD \) (the one that contains the points \( B \) and \( C \)) contains \( 2n-1-k \) points from \( P \) in its interior. At least \( n \) points from \( P \) are between \( D \) and \( B \) on the arc that contains \( C \). Hence, there are \( n-1-k \) choices for \( B \) and, analogously, \( n-1-k \) choices for \( C \).

The required number of \( m \)-gons with \( A = A_1 \) is therefore

\[
\sum_{k=1}^{n-2} (n-1-k)^2 \binom{k}{m-4} = \sum_{j=1}^{n-2} j^2 \binom{n-1-j}{m-4}.
\]

Dropping the hypothesis \( A = A_1 \), we get

\[
(2n+1) \left[ \sum_{j=1}^{n-2} j^2 \binom{n-1-j}{m-4} \right]
\]

regular polygons. To complete the proof, we prove that

\[
\sum_{j=1}^{n-2} j^2 \binom{n-1-j}{m-4} = \binom{n}{m-1} + \binom{n+1}{m-1}.
\]  

We prove it by mathematical induction. It holds for \( n = m + 1 \), because

\[
\sum_{j=1}^{m-1} j^2 \binom{m-j}{m-4} = \binom{m-1}{3} + 4 \binom{m-2}{2} + 9(m-3) + 16 = \binom{m+1}{m-1} + \binom{m+2}{m-1}.
\]
The relation (1) also holds for $m = 5$:

\[
\sum_{j=1}^{n-2} j^2(n - 1 - j) = \left[ (n - 1) \sum_{j=1}^{n-2} j^2 \right] - \left[ \sum_{j=1}^{n-2} j^3 \right] \\
= \frac{(n - 2)(n - 1)^2(2n - 3)}{6} - \frac{(n - 2)^2(n - 1)^2}{4} \\
= \frac{(n - 2)(n - 1)^2}{24} (8n - 12) - (6n - 12) \\
= \frac{2n(n - 2)(n - 1)^2}{24} \\
= \frac{n(n - 2)(n - 1)}{24} (2n - 2) \\
= \frac{n(n - 2)(n - 1)}{24} ((n - 3) + (n + 1)) \\
= \frac{n}{4} + \frac{n + 1}{4}.
\]

The induction step from $(m - 1, n)$ and $(m, n)$ to $(m, n + 1)$ is

\[
\sum_{j=1}^{n-2} j^2 \left( \frac{n - j}{m - 4} \right) = \sum_{j=1}^{n-2} j^2 \left( \frac{n - 1 - j}{m - 4} \right) + \sum_{j=1}^{n-2} j^2 \left( \frac{n - 1 - j}{m - 4} \right) \\
= \frac{n}{m - 1} + \frac{n + 1}{m - 1} + \frac{n}{m - 2} + \frac{n + 1}{m - 2} \\
= \frac{n + 1}{m - 1} + \frac{n + 2}{m - 1}.
\]

This completes the induction and therefore the proof of (1).

**OC35.** Find all pairs of integers $(x, y)$ such that

\[ y^3 = 8x^6 + 2x^3y - y^2. \]

*(Originally question #3 from 2009 Italian Team Selection Test.)*

*Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA and Titu Zvonaru, Comănești, Romania. We give the solution of Manes.*

We show that the only solutions are $(0, -1), (0, 0)$ and $(1, 2)$. It is easy to check that these are solutions, we show now that there is no other solution.

Solving the quadratic equation in $x^3$ we get

\[ x^3 = \frac{-y \pm \sqrt{9 + 8y}}{8}. \]

From here we get $y \geq -1$. If $y = -1$ then $x = 0$ while if $y = 0$ we get that $x = 0$. 

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Assume \( y \geq 1 \). Since \( x, y \) are integers, it follows that \( \sqrt{9 + 8y} \) is rational thus integer. Moreover, as \( 9 + 8y \) is odd, \( \sqrt{9 + 8y} \) must be an odd integer.

Write
\[
\sqrt{9 + 8y} =: 2n + 3.
\]
Then \( n \geq 1 \),
\[
y = \frac{n(n + 3)}{2},
\]
and either
\[
x^3 = \frac{n(n + 1)(n + 3)}{8}
\]
or
\[
x^3 = -\frac{n(n + 2)(n + 3)}{8}.
\]
If \( n = 1 \) then \( y = 2 \) and \( x = 1 \), while for \( n \geq 2 \) we have
\[
(n + 1)^3 < n(n + 1)(n + 3) < (n + 2)^3 \text{ and }
\]
\[
-(n + 2)^3 < -n(n + 2)(n + 3) < -(n + 1)^3.
\]
Thus
\[
\frac{n + 1}{2} < \sqrt[3]{\frac{n(n + 1)(n + 3)}{8}} < \frac{n + 2}{2},
\]
\[
-\frac{n + 2}{2} < \sqrt[3]{-\frac{n(n + 2)(n + 3)}{8}} < -\frac{n + 1}{2},
\]
and hence \( x \) cannot be an integer. This shows that there are no solutions for \( n \geq 2 \).