MAYHEM SOLUTIONS

Mathematical Mayhem is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of Crux will appear in this volume, after which time Mathematical Mayhem will be discontinued in Crux. New Mayhem problems will appear when the journal is relaunched in 2013.

M495. Proposed by the Mayhem Staff.

All possible lines are drawn through the point (0,0) and the points (x,y), where x and y are whole numbers with 1 ≤ x, y ≤ 10. How many distinct lines are drawn?

Solution by Florencio Cano Vargas, Inca, Spain.

Since the lines pass by (0,0), each line is characterized by a single parameter: the slope m and counting the lines amounts to counting all possible values of the slope. Since \( m = \frac{y}{x} \), the different values of the slope are the irreducible fractions \( \frac{y}{x} \) with 1 ≤ x, y ≤ 10. The boundaries are \( \frac{1}{10} ≤ m ≤ \frac{10}{1} \).

Let us call \( N \) the number of lines, \( N[m < 1] \) the number of lines with \( m < 1 \) and \( N[m > 1] \) the number of lines with \( m > 1 \). By symmetry around the line \( x = y \) (i.e. \( m = 1 \)), we have \( N[m < 1] = N[m > 1] \) and the requested number can be written as:

\[
N = 2N[m < 1] + 1
\]

where the last term accounts for the case \( m = 1 \) which is considered separately.

To evaluate \( N[m < 1] \) we still have to count the number of irreducible fractions \( \frac{y}{x} \) with 1 ≤ y < x ≤ 10. We study the different cases for x:

- \( x = 1 \). This case gives no values of \( m < 1 \).
- \( x = 2 \). We look for irreducible fractions \( \frac{y}{2} \), i.e., values of y relatively prime to 2, which is just \( y = 1 \).
- \( x = 3 \). We look for irreducible fractions \( \frac{y}{3} \), i.e., values of y relatively prime to 3, which are two values \( y = 1, 2 \).

From these cases it can be inferred that we look for the number of values of y relatively prime to x and which are smaller than x. This is just Euler’s totient function \( \varphi(x) \) (Euler’s totient function \( \varphi(n) \) is defined as the number of positive integers less than or equal to n that are relatively prime to n. We are looking for the number of positive integers strictly less than n which are relatively prime to n, but this subtlety makes no difference since a number is not relatively prime to itself.) Then we can write:

\[
N[m < 1] = \sum_{x=2}^{10} \varphi(x) = 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 + 4 = 31
\]

and the final answer is \( N = 2 \cdot 31 + 1 = 63 \) different lines.

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M496. Proposed by Sally Li, student, Marc Garneau Collegiate Institute, Toronto, ON.

Show that if we write the numbers from 1 to \(n\) around a circle, in any order, then, for all \(x = 1, 2, \ldots, n\), we are guaranteed to find a block of \(x\) consecutive numbers that add up to at least \(\lceil x(n + 1) \rceil \cdot \frac{n(n + 1)}{2}\). Here \(\lceil y \rceil\) is the ceiling function, that is, the least integer greater than or equal to \(y\). So \(\lceil 6.2 \rceil = 7\), \(\lceil \pi \rceil = 4\), \(\lceil -8.3 \rceil = -8\) and \(\lceil 10 \rceil = 10\).

Solution by Florencio Cano Vargas, Inca, Spain.

Let us first note that the sum of all the numbers of the circle is given by \(\frac{n(n + 1)}{2}\).

Let \(x < n\) be the number of consecutive numbers we take at a time. For a given \(x\) we have \(n\) different combinations of consecutive numbers and each number of the circle enters in \(x\) combinations. If we define \(S_k\) as the sum of the numbers in the \(k\)th combination \((k = 1, \ldots, n)\), then the sum of the numbers contained in all the combinations \(\{S_1, S_2, \ldots, S_n\}\) is \(x\) times the sum of the numbers of the circle:

\[
S_1 + S_2 + \cdots + S_n = x \cdot \frac{n(n + 1)}{2}
\]

hence

\[
\frac{n}{n} S_1 + S_2 + \cdots + S_n = \frac{x(n + 1)}{2}.
\]

Then the problem is equivalent to prove that in an arithmetic mean of positive integers, there is at least one which is larger than or equal to the mean. Let us suppose that it is not so, and that all \(S_k\) satisfy:

\[
S_k < \frac{x(n + 1)}{2}, \quad k = 1, \ldots, n,
\]

but then

\[
S_1 + S_2 + \cdots + S_n < x \cdot \frac{n(n + 1)}{2},
\]

which contradicts (1). Therefore, there must be at least some \(S_k \geq \frac{x(n+1)}{2}\). Since \(S_k\) is integer for any \(k\), but \(\frac{x(n+1)}{2}\) isn’t necessarily an integer, then we can strengthen the claim, that is, there will be at least one \(k\) for which

\[
S_k \geq \lceil \frac{x(n + 1)}{2} \rceil.
\]

It remains to prove the case \(x = n\). In that case, the only block is the whole set of numbers, whose sum is \(\frac{n(n+1)}{2} = \lceil \frac{x(n+1)}{2} \rceil\), which completes the proof.

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M497. Proposed by Pedro Henrique O. Pantoja, student, UFRN, Brazil.

Find all integers $a$, $b$, $c$ where $c$ is a prime number such that $a^b + c$ and $a^b - c$ are both perfect squares.

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

Assume $c$ is an odd prime and $a^b + c = m^2$ and $a^b - c = n^2$ for some integers $m$ and $n$. Subtracting the two equations yields $2c = m^2 - n^2 = (m + n)(m - n)$. Unique factorization then implies $m - n = 2$ and $m + n = c$, which are contradictory equations since $m \pm n$ always have the same parity. Hence, if $c$ is an odd prime, there are no integers $a$, $b$ such that $a^b + c$ and $a^b - c$ are both perfect squares.

However, if $c = 2$, then $a^b + c = m^2$ and $a^b - c = n^2$ imply $m^2 - n^2 = (m+n)(m-n) = 4$. Therefore, either $m+n = 4$ and $m-n = 1$ or $m+n = m-n = 2$ by unique factorization. As noted above, the equations $m+n = 4$ and $m-n = 1$ are contradictory. If $m+n = m-n = 2$, then $m = 2$ and $n = 0$. Therefore $a^b = 2$, whence $a = 2$ and $b = 1$. Thus, $(a, b, c) = (2, 1, 2)$ is the unique solution.

M498. Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John’s, NL.

Right triangle $ABC$ has its right angle at $C$. The two sides $CB$ and $CA$ are of integer length. Determine the condition for the radius of the incircle of triangle $ABC$ to be a rational number.

Solution by Cäsio dos Santos Sousa, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil.

Take $BC = a$, $AC = b$, and $AB = c$. To calculate the radius $r$ of the incircle of triangle $ABC$, we may use the figure below.

As the incenter is the intersection of the angle bisectors, there will be three pairs of congruent triangles formed (one shaded from each pair), each can be proved using $AAS$ comparison (one of the angles comes from the bisection, the other angle is $90^\circ$, and the common side has length $r$). Then, from the triangles...
formed with vertex $A$:

$$b - r = c - (a - r)$$
$$r = \frac{a + b - c}{2}.$$  

We were given that $a$ and $b$ are integers, hence, if we want a rational value for $r$, then $c$ must be rational. The Pythagorean theorem gives $c^2 = a^2 + b^2$, an integer, so $c$ is either integer or irrational. If we want $r$ to be rational, then $c$ must be an integer.

To find $a$ and $b$ such that $\sqrt{a^2 + b^2}$ will be an integer value, we may use Euclid’s Formula:

$$a = k(m^2 - n^2)$$
$$b = k(2mn)$$
$$c = k(m^2 + n^2)$$

If $k, m$ and $n$ are integers, and $m > n$, then $(a, b, c)$ will be a Pythagorean triple, and then $r$ will be a rational value.

Also solved by KONSTANTINOS DAGIADAS, Agrinio, Greece; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICARD PEIRÓ, IES “Abastos”, Valencia, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer.

**M499. Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.**

Two circles of radius 1 are drawn so that each circle passes through the centre of the other circle. Find the area of the goblet like region contained between the common radius, the circumferences and one of the common tangents as shown in the diagram to the right.

**Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.**

Let $A$, $B$, $C$, and $D$ be the points as shown in the figure. Let $F$ be the darkly shaded area, that is, half of the upper part of the goblet. Triangles $ABC$ and $ABD$ are equilateral, since each side is a radius of one of the circles. By symmetry, the second half of the upper part of the goblet, the part beside $F$, is equal to the shaded area above $F$.

Hence, the area of the upper part of the goblet, $2F$, is equal to the area of a 120° sector, minus the area of triangle $ACD$.  

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Note that triangles $AEC$, $BEC$, $AED$ and $BED$ are congruent by hypotenuse-side, hence the $|ACD| = |AEC| + |AED| = |AEC| + |BEC| = |ABC|$ (where $|XYZ|$ represents the area of triangle $XYZ$). Thus

$$2F = \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

Let $E$, $H$ and $I$ be the points as shown in the diagram.

Let $G$ be the darkest shaded area, that is, half of the lower part of the goblet. Looking at the rectangle $BEHI$ with sides 1 and $\frac{1}{2}$, it is broken into three parts: half of the base of the goblet ($G$), triangle $BED$ and sector $BDI$. The triangle is half of an equilateral triangle, and the sector has angle $30^\circ$, hence

$$G = \frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}.$$

Thus the area of the goblet is

$$2F + 2G = \frac{\pi}{3} - \frac{\sqrt{3}}{4} + 2 \left( \frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12} \right) = 1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2} \approx 0.6576.$$

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; ANDHIFA GILANG, student, SMPN 8, Yogyakarta, Indonesia; THARIQ SURYA GUMELAR, student, SMPN 8, Yogyakarta, Indonesia; RICARD PEIRÓ, IES “Abastos”, Valencia, Spain; ALMER FANDRIYANTO, student, SMAN 25, Bandung, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One correct solution with no name on it and one incorrect solution were also received.

**M500. Proposed by Eduard T.H. Wang and Dexter S.Y. Wei, Wilfrid Laurier University, Waterloo, ON.**

Let $\mathbb{N}$ denote the set of natural numbers.

(a) Show that if $n \in \mathbb{N}$, there do not exist $a, b \in \mathbb{N}$ such that $\frac{[a, b]}{a + b} = n$, where $[a, b]$ denotes the least common multiple of $a$ and $b$.

(b) Show that for any $n \in \mathbb{N}$, there exist infinitely many triples $(a, b, c)$ of natural numbers such that $\frac{[a, b, c]}{a + b + c} = n$, where $[a, b, c]$ denotes the least common multiple of $a, b$ and $c$.

**Solution by Florencio Cano Vargas, Inca, Spain.**

(a) Let us suppose that for a given $n \in \mathbb{N}$, there exist $a, b \in \mathbb{N}$ that satisfy the condition given in the problem. We can write $a = da', b = db'$ where $d = \gcd(a, b)$. Then $\gcd(a', b') = 1$ and $[a, b] = da'b'$ so the condition of the problem can be rewritten as:

$$\frac{a'b'}{a' + b'} = n.$$
First of all note that since \( a' \) and \( b' \) are relatively prime and we cannot have \( a' + b' = 1 \), then \( a' + b' \neq 1 \) must divide \( a'b' \).

Let \( p > 1 \) be any prime common factor of \( a'b' \) and \( a' + b' \). Since \( a' \) and \( b' \) are relatively prime they don’t share any prime factor, and therefore \( p \) is a factor either of \( a' \) or \( b' \). Let us assume without loss of generality that it is a factor of \( a' \), i.e. \( a' = ps \) for some integer \( s \). Then for some integer \( q \)

\[
a' + b' = pq \Rightarrow ps + b' = pq \Rightarrow b' = p(q - s)
\]

and then \( p \) is also a factor of \( b' \) which contradicts the fact that \( \gcd(a', b') = 1 \).

This means that \( a'b' \) and \( (a' + b') \) are relatively prime and then the fraction \( \frac{a'b'}{a' + b'} \) is irreducible and \( n \not\in \mathbb{N} \), which contradicts the initial assumption.

(b) We can look for triples \((a, b, c)\) such that \( a = a'd, b = b'd, c = c'd \) with \( \gcd(a, b, c) = d \) and with \( a', b', c' \) pairwise relatively prime. Then we can write \( \text{lcm}(a, b, c) = da'b'c' \) and the condition of the problem can be rewritten as:

\[
\frac{a'b'c'}{a' + b' + c'} = n.
\]

To enforce this property let us choose \( b' = a' + 1 \), which is always relatively prime with \( a' \) and \( c' = 1 \). We end up with a condition for \( a' \):

\[
\frac{a'(a' + 1)}{a' + (a' + 1) + 1} = n \iff \frac{a'(a' + 1)}{2(a'+1)} = n \iff a' = 2n
\]

which gives \( b' = 2n + 1 \) and \( c' = 1 \). Hence a solution for a given \( n \in \mathbb{N} \) is the infinite set of triples:

\[(a, b, c) = (2nd, 2nd + d, d), \ d \in \mathbb{N}.
\]

Also solved by DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposers.