SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3492 $\star$. [2009 : 515, 518; 2010 : 558] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let P be a point in the interior of tetrahedron $ABCD$ such that each of $\angle PAB, \angle PBC, \angle PCD$, and $\angle PDA$ is equal to $\arccos\sqrt{\frac{2}{3}}$. Prove that $ABCD$ is a regular tetrahedron and that P is its centroid.

Solution by Tomasz Cieśla, student, University of Warsaw, Poland.

The claim is not true; we shall see that there are infinitely many counterexamples. Consider points A', B, C on a circle with centre P such that $\angle PA'B = \angle PBA' = \angle PBC = \angle PCB = \arccos\sqrt{\frac{2}{3}}$. For $\alpha \in (0, \pi)$ the rotation about line PB through the angle α maps point A' into a point A (preserving angles $PA'B$ and PBA'). Let D be the reflection of point B in the plane PCA (so that $\angle PCD = \angle PCB$ and $\angle PDA = \angle PBA = \angle PBA'$.) Then the tetrahedron $ABCD$ satisfies

$$\angle PAB = \angle PBC = \angle PCD = \angle PDA = \arccos\sqrt{\frac{2}{3}},$$

as required. Note that we can choose α so that $ABCD$ is not regular; in fact, there is only one value of α which produces a regular tetrahedron (occurring when $\angle PAC = \arccos\sqrt{\frac{2}{3}}$). Values of α close to that ensure that P lies in the interior of $ABCD$ and, therefore, provide counterexamples to the problem as it was stated.

No other correspondence about this problem has been received.

3631. [2011: 171, 173] *Proposed by Michel Bataille, Rouen, France.*

Let $\{x_n\}$ be the sequence satisfying $x_0 = 1$, $x_1 = 2011$, and $x_{n+2} = 2012x_{n+1} - x_n$ for all nonnegative integers n. Prove that

$$\frac{(2010 + x_n^2 + x_{n+1}^2)(2010 + x_{n+2}^2 + x_{n+3}^2)}{(2010 + x_{n+1}^2)(2010 + x_{n+2}^2)}$$

is independent of n.

Solution by Arkady Alt, San Jose, CA, USA.

More generally, let a be an integer and let $\{x_n\}$ be determined by $x_{-1} = x_0 = 1$, $x_1 = a + 1$ and $x_{n+2} = (a + 2)x_{n+1} - x_n$ for $n \geq 0$. Since

$$\begin{aligned} x_n x_{n+2} - x_{n+1}^2 &= x_n[(a+2)x_{n+1} - x_n] - x_{n+1}^2 = x_{n+1}[(a+2)x_n - x_{n+1}] - x_n^2 \\ &= x_{n-1}x_{n+1} - x_n^2, \end{aligned}$$

it follows that

$$x_nx_{n+2} - x_{n+1}^2 = x_0x_2 - x_1^2 = (a^2 + 3a + 1) - (a + 1)^2 = a$$

for $n \geq 0$. Therefore, for $n \geq 0$,

$$\begin{aligned}\frac{(a + x_n^2 + x_{n+1}^2)(a + x_{n+2}^2 + x_{n+3}^2)}{(a + x_{n+1}^2)(a + x_{n+2}^2)} &= \frac{(x_{n-1}x_{n+1} + x_{n+1}^2)(x_{n+1}x_{n+3} + x_{n+3}^2)}{(x_nx_{n+2})(x_{n+1}x_{n+3})} \\ &= \frac{x_{n+1}(x_{n-1} + x_{n+1})x_{n+3}(x_{n+1} + x_{n+3})}{(x_{n+1}x_{n+3})(x_nx_{n+2})} \\ &= \frac{[(a+2)x_n][(a+2)x_{n+2}]}{x_nx_{n+2}} = (a+2)^2.\end{aligned}$$

Taking $a = 2010$ yields the value 2012^2 for the expression in the problem.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; KEE-WAI LAU, Hong Kong, China; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comǎneşti, Romania; and the proposer. Apostolopoulos and the proposer had solutions similar to the one given, while the remaining solvers solved the recursion and used the formula for the general term. One additional person simply gave the answer with no justification.

3632. [2011: 171, 173] *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let k be a real number such that $0 \leq k \leq 56$. Prove that the equation below has exactly two real solutions:

$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \;=\; k(x^2 - 7x) + 720\,.$$

Solution by Richard I. Hess, Rancho Palos Verdes, CA, USA.

We prove that the result holds as long as $k < 945/16 = 59.0625$. The difference $P(x)$ betweeen the two sides of the equation is given by

$$\begin{aligned}P(x) &= [(x-1)(x-6)][(x-2)(x-5)][(x-3)(x-4)] - k(x^2 - 7x) - 720 \\ &= [(x^2 - 7x) + 6][(x^2 - 7x) + 10][(x^2 - 7x) + 12] - k(x^2 - 7x) - 720 \\ &= (x^2 - 7x)[(x^2 - 7x)^2 + 28(x^2 - 7x) + (252 - k)] \\ &= \frac{1}{16}(x^2 - 7x)\{[(2x-7)^2 - 49]^2 + 112[(2x-7)^2 - 49] + (4032 - 16k)\} \\ &= \frac{1}{16}x(x-7)[(2x-7)^4 + 14(2x-7)^2 + (945 - 16k)].\end{aligned}$$

When $k < 945/16$, the final factor is positive for all real x, so that the only real roots of $P(x)$ are 0 and 7.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, *USA; ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Cománeşti, Romania; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer.*

It is easy to determine the situation for other values of k. When $k = 945/16$, then $x = 7/2$ is an additional root of $P(x)$. When $k > 945/16$, then the final factor vanishes for one positive and one negative value of $(2x-7)^2$. The positive value gives rise to two real roots of the final factor. However, these two roots are 0 and 7 when $k = 252$. Thus, $P(x)$ has two roots if and only if $k < 945/16$ or $k = 252$, three roots when $k = 945/16$ and four roots when $k \neq 252$ and $k > 945/16$.

About half of the solvers identified values of k less than 945/16 as yielding two solutions, but only four picked up $k = 252$. Nine solvers employed the substitution $y = x^2 - 7x$ which led to their analyzing the equation $y(y^2 + 28y + 252 - k) = 0$, while Geupel and Stadler let $u = \frac{x+7}{2}$ and analyzed the equation $(u^2 - 49)(u^4 + 14u^2 + 945 - 16k) = 0$.

3633. [2011 : 171, 173] *Proposed by Razvan Tudoran, Universitatea de Vest din Timisoara, Timisoara, Romania; and Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $g_1(x) = x$ and for natural numbers $n > 1$ define $g_n(x) = x^{g_{n-1}(x)}$. Let $f : (0,1) \to \mathbb{R}$ be the function defined by $f(x) = g_n(x)$, where $n = \left\lfloor \frac{1}{x} \right\rfloor$. For example, $f\left(\frac{1}{3}\right) = \frac{1}{3}^{\frac{1}{3}^{\frac{1}{3}}}$. Here $\lfloor a \rfloor$ denotes the floor of a. Determine $\lim\limits_{x\to 0^+} f(x)$ or prove it does not exist.

[*Ed.: Note when the problem was published, Razvan Turdoran's name was mistakenly omitted from the problem. Our apologies to Razvan.*]

Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy, modified slightly by the editor.

The limit does not exist. We consider $\lim\limits_{x\to 0^+} f(x)$ when $x = \dfrac{1}{2n+1}$ and $x = \dfrac{1}{2n}$ separately where $n \in \mathbb{N}$.

In the published solution to problem #922 of the *College Mathematics Journal* (Vol. 42, No. 2, March 2011; pp. 152-155) the following results were obtained:

(a) $$\frac{1}{2n+1} < f\left(\frac{1}{2n+1}\right) < \frac{1}{\ln(2n+1)},$$

(b) $$\left(\frac{1}{2n}\right)^{\frac{1}{\ln(2n)}} < f\left(\frac{1}{2n}\right) < \left(\frac{1}{2n}\right)^{\frac{1}{2n}}.$$

From (a) $\lim\limits_{n\to\infty} f\left(\dfrac{1}{2n+1}\right) = 0$ follows immediately. Hence, if $\lim\limits_{x\to 0^+} f(x)$ exists, then letting $n = \left\lfloor \dfrac{1}{x} \right\rfloor$ we must have $\lim\limits_{n\to\infty} f\left(\dfrac{1}{2n}\right) = 0$ which is impossible in view of (b) since $\lim\limits_{n\to\infty}\left(\frac{1}{2n}\right)^{\frac{1}{2n}}=1$ [*Ed: This can be shown easily by using the l'Hôpital's rule.*] and $\left(\frac{1}{2n}\right)^{\frac{1}{\ln(2n)}}=e^{-1}>0$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and the proposers.

The proposers of the current problem are the same as the problem in the College Mathematics Journal. They simply combine the results in (a) and (b) and came to the immediate conclusion about $\lim\limits_{x\to 0^+} f(x)$.

3634. [2011: 171, 174] *Proposed by Michel Bataille, Rouen, France.*

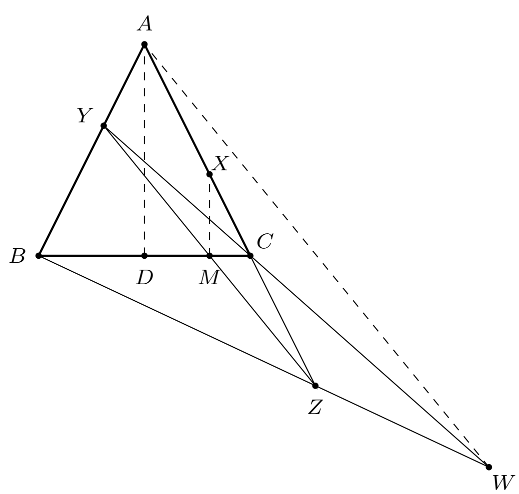
ABC is an isosceles triangle with $AB=AC$. Points X, Y and Z are on rays $\overrightarrow{AC}$, $\overrightarrow{BA}$ and $\overrightarrow{AC}$ respectively with $AZ>AC$ and $AX=BY=CZ$.

(a) Show that the orthogonal projection of X onto BC is the midpoint of YZ.

(b) If BZ and YC intersect in W, show that the triangles CYA and CWZ have the same area.

Composition of solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Titu Zvonaru, Cománeşti, Romania.

First, we will assume that $X\neq C$, as otherwise the problem becomes trivial. Let D be the midpoint of BC, and let YZ and BC intersect in M.



a) By Menelaus' Theorem applied to $\triangle AYZ$ and the transversal $B-M-C$, we have: $\frac{AB}{BY}\cdot\frac{YM}{MZ}\cdot\frac{ZC}{CA}=1$ so $\frac{YM}{MZ}=1$; hence M is the midpoint of YZ. Similarly, Menelaus' Theorem applied to $\triangle ABC$ and the transversal $Z - M - Y$ yields:

$$\frac{AY}{YB}\cdot\frac{BM}{MC}\cdot\frac{CZ}{ZA} = 1 \;\Leftrightarrow\; \frac{BM}{MC} = \frac{ZA}{AY} = \frac{AC+ZC}{AC-ZC}$$
$$\Leftrightarrow\; \frac{BM+MC}{MC} = \frac{2AC}{AC-AX}$$
$$\Leftrightarrow\; \frac{2DC}{MC} = \frac{2AC}{XC} \;\Leftrightarrow\; \frac{DC}{MC} = \frac{AC}{XC}.$$

Hence $XM \parallel AD$, which proves the result that the midpoint M is the orthogonal projection of X onto BC.

b) By Ceva's Theorem applied to $\triangle YBZ$ and the point C, we have:

$$\frac{YA}{AB}\cdot\frac{BW}{WZ}\cdot\frac{ZM}{MY} = 1.$$

Since M is the midpoint of YZ, we conclude that $\frac{YA}{AB} = \frac{ZW}{WB}$, which shows that $AW \parallel YZ$. Now, if $[ABC]$ denotes the area of $\triangle ABC$, then we have

$$[CYA] = [AYZ] - [CYZ] = [WYZ] - [CYZ] = [CWZ],$$

as desired.

Also solved by RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; and the proposer.

3635. [2011 : 172, 174] *Proposed by Mehmet Sahin, Ankora, Turkey.*

Let ABC be an acute-angled triangle with circumradius R, inradius r, semiperimeter s, and with points $A' \in BC, B' \in CA$, and $C' \in AB$ arranged so that

$$\angle ACC' = \angle CBB' = \angle BAA' = 90^\circ.$$

Prove that:

(a) $|BC'||CA'||AB'| = abc$;

(b) $\dfrac{|AA'|}{|BC'|}\dfrac{|BB'|}{|CA'|}\dfrac{|CC'|}{|AB'|} = \tan A \tan B \tan C$;

(c) $\dfrac{Area(A'B'C')}{Area(ABC)} = \dfrac{4R^2}{s^2-(2R+r)^2} - 1.$

Solution by Titu Zvonaru, Comănești, Romania.

(a) From the right-angled triangle $AC'C$ we have

$$\cos A = \frac{AC}{AC'} = \frac{b}{c+BC'} = \frac{a\cos C + c\cos A}{c+BC'},$$

whence

$$BC' = \frac{a\cos C}{\cos A}.$$

Similarly $CA' = \dfrac{b\cos A}{\cos B}$ and $AB' = \dfrac{c\cos B}{\cos C}$, and the equation in (a) follows immediately.

(b) In right triangles ABA', BCB', and CAC' we have $AA' = c\tan B, BB' = a\tan C$, and $CC' = b\tan A$, respectively. Together with part (a) the product of these line segments produces the equality of part (b).

(c) Let us use square brackets to denote areas. Since $\angle C'BA' = 180^\circ - B$, we have (with the help of part (a))

$$\begin{aligned}[A'BC'] = \frac{BC' \cdot BA' \cdot \sin\angle C'BA'}{2} &= \frac{a\cos C}{\cos A} \cdot \frac{c}{\cos B} \cdot \frac{\sin B}{2} \\ &= \frac{ac\sin B}{2} \cdot \frac{\cos C}{\cos A\cos B} = [ABC]\frac{\cos C}{\cos A\cos B}.\end{aligned}$$

With the analogous results for triangles $B'CA'$ and $C'AB'$ we have

$$\begin{aligned}[A'B'C'] &= [ABC] + [A'BC'] + [B'CA'] + [C'AB'] \\ &= [ABC] \cdot \left(1 + \frac{\cos C}{\cos A\cos B} + \frac{\cos A}{\cos B\cos C} + \frac{\cos B}{\cos A\cos C}\right) \\ &= [ABC] \cdot \left(1 + \frac{\cos^2 A + \cos^2 B + \cos^2 C}{\cos A\cos B\cos C}\right).\end{aligned}$$

From $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$ the final equation becomes

$$\frac{[A'B'C']}{[ABC]} = \frac{1}{\cos A\cos B\cos C} - 1.$$

The problem has now been reduced to proving that

$$\frac{1}{\cos A\cos B\cos C} = \frac{4R^2}{s^2 - (2R+r)^2}, \tag{1}$$

but a proof of this formula can be found in [2, p. 56, formula 37] and [1, formula 205].

[*Ed.* The references were provided by Arslanagić and by Bellot; instead of supplying a reference, Zvonaru used the sine law to reduce equation (1) to a better-known equation.]

References

[1] Anonymous, *Relations entre les éléments d'un triangle: Recueil de 273 formules relatives au triangle avec leurs démonstrations*, Paris, Librairie Nony & Cie, 1893.

[2] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; and the proposer.

3636. [2011 : 172, 174] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let a, b, c, and d be nonnegative real numbers such that $a+b+c+d=2$. Prove that

$$ab(a^2+b^2+c^2)+bc(b^2+c^2+d^2)+cd(c^2+d^2+a^2)+da(d^2+a^2+b^2) \;\le\; 2\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $x=a+c$ and $y=b+d$. Then $x+y=2$, and we have

$$\begin{aligned}\sum_{\text{cyclic}} ab(a^2+b^2+c^2) &\le (ab+bc+cd+da)(a^2+b^2+c^2+d^2+2ac+2bd)\\ &= (a+c)(b+d)\left((a+c)^2+(b+d)^2\right) = x^3y+xy^3\\ &= \frac{1}{8}\left((x+y)^4-(x-y)^4\right) \le \frac{1}{8}(x+y)^4 = 2.\end{aligned}$$

The example $(a,b,c,d)=(1,1,0,0)$ shows that the inequality is sharp.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

From the proof featured above it is easy to see that equality holds if and only if $(a,b,c,d)=(1,1,0,0)$ *or* $(0,1,1,0)$ *or* $(0,0,1,1)$ *or* $(1,0,0,1)$. *This was explicitly pointed out by AN-anduud problem solving group and Arslanagić.*

3637. [2011: 172, 174] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let x be a real number with $|x|<1$. Determine

$$\sum_{n=1}^{\infty}(-1)^{n-1}n\left(\ln(1-x)+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n}\right)\,.$$

I. Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Observe that

$$\begin{aligned} S'(x) &= \sum_{n=1}^{\infty}(-1)^{n-1}n\left(\frac{-1}{1-x}+1+x+x^2+\cdots+x^{n-1}\right) \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}n\left(\frac{-1}{1-x}+\frac{1-x^n}{1-x}\right) = \frac{-x}{1-x}\sum_{n=1}^{\infty}(-1)^{n-1}nx^{n-1} \\ &= \frac{-x}{(1-x)(1+x)^2} = \frac{1}{4}\left[\frac{2}{(1+x)^2}-\frac{1}{1+x}-\frac{1}{1-x}\right]. \end{aligned}$$

Noting that $S(0)=0$, we deduce that

$$\begin{aligned} S(x) &= \frac{1}{4}\left[2-\frac{2}{1+x}-\ln(1+x)+\ln(1-x)\right] \\ &= \frac{1}{4}\left[\frac{2x}{1+x}+\ln\frac{1-x}{1+x}\right] = \frac{x}{2(1+x)}+\frac{1}{4}\ln\frac{1-x}{1+x}. \end{aligned}$$

Editor's comment: We can also write

$$S'(x) = \frac{1}{2}\left[\frac{x}{(1+x)^2}-\frac{1}{1-x^2}\right],$$

which leads to $S(x)=\frac{1}{2}[x(1+x)^{-1}-\tanh^{-1}x]$.

II. Solution by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

$$\begin{aligned} S(x) &= -\sum_{n=1}^{\infty}(-1)^{n-1}n\left(\sum_{k=n+1}^{\infty}\frac{x^k}{k}\right) \\ &= -\sum_{n=2}^{\infty}\frac{x^n}{n}\sum_{j=1}^{n-1}(-1)^{j-1}j = -\sum_{n=2}^{\infty}\frac{x^n}{n}(-1)^n\left\lfloor\frac{n}{2}\right\rfloor \\ &= \sum_{n=2}^{\infty}(-1)^{n+1}\left\lfloor\frac{n}{2}\right\rfloor\frac{x^n}{n} = -\frac{1}{2}\sum_{k=1}^{\infty}x^{2k}+\sum_{k=1}^{\infty}\frac{kx^{2k+1}}{2k+1} \\ &= -\frac{1}{2}\sum_{k=1}^{\infty}x^{2k}+\frac{1}{2}\sum_{k=1}^{\infty}x^{2k+1}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{2k+1}}{2k+1} \\ &= -\frac{1}{2}\frac{x^2(1-x)}{1-x^2}-\frac{1}{4}\ln\frac{1+x}{1-x}+\frac{x}{2} \\ &= \frac{1}{2}\left[\frac{-x^2}{1+x}+x\right]+\frac{1}{4}\ln\frac{1-x}{1+x} = \frac{x}{2(1+x)}+\frac{1}{4}\ln\frac{1-x}{1+x}. \end{aligned}$$

III. Solution by Michel Bataille, Rouen, France.

For $|x| < 1$, we use the Taylor representation

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n dt$$

with $f(x) = \ln(1-x)$ to obtain

$$\begin{aligned} \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} + \int_0^x \frac{(x-t)^n}{n!} \cdot \frac{-n!}{(1-t)^{n+1}} dt \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \int_0^x \frac{u^n}{1-u} du, \end{aligned}$$

where the two integrals are related by the substitution $(u-1)t = u-x$. Therefore

$$\begin{aligned} S(x) &= \sum_{n=1}^{\infty} \int_0^x \frac{(-1)^n n u^n}{1-u}\, du = \int_0^x \frac{1}{1-u}\left(\sum_{n=1}^{\infty}(-1)^n n u^n\right) du \\ &= \int_0^x \frac{-u}{(1-u)(1+u)^2}\, du = \frac{1}{4}\int_0^x \left[\frac{2}{(1+u)^2} - \frac{1}{1-u} - \frac{1}{1+u}\right] du \\ &= \frac{x}{2(1+x)} + \frac{1}{4}\ln\frac{1-x}{1+x}. \end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANASTASIOS KOTRONIS, Athens, Greece; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3638. [2011: 234, 237] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle and let points D, E, F lie on lines BC, CA, AB, respectively, such that

$$BD : DC = \lambda : 1-\lambda, \quad CE : EA = \mu : 1-\mu, \quad AF : FB = \nu : 1-\nu.$$

Show that DEF is a pedal triangle with regard to ΔABC if and only if

$$(2\lambda - 1)BC^2 + (2\mu - 1)CA^2 + (2\nu - 1)AB^2 = 0.$$

Solution by the proposer.

Let A', B', and C' be the midpoints of BC, CA, and AB, respectively. Since $\overrightarrow{BD} = \lambda\overrightarrow{BC}$ and $\overrightarrow{CD} = (\lambda - 1)\overrightarrow{BC}$, we have $(2\lambda - 1)\overrightarrow{BC} = 2\overrightarrow{A'D}$. Similarly, $(2\mu - 1)\overrightarrow{CA} = 2\overrightarrow{B'E}$ and $(2\upsilon - 1)\overrightarrow{AB} = 2\overrightarrow{C'F}$ so that the given condition is equivalent to

$$\overrightarrow{A'D} \cdot \overrightarrow{BC} + \overrightarrow{B'E} \cdot \overrightarrow{CA} + \overrightarrow{C'F} \cdot \overrightarrow{AB} = 0. \tag{1}$$

Now, suppose that D, E, F are the orthogonal projections of some point P onto BC, CA, AB, respectively. Then,

$$\overrightarrow{A'D}\cdot\overrightarrow{BC}=\overrightarrow{A'P}\cdot\overrightarrow{BC}=-\frac{1}{2}(\overrightarrow{PB}+\overrightarrow{PC})\cdot(\overrightarrow{PC}-\overrightarrow{PB})=\frac{1}{2}(PB^2-PC^2)$$

and similarly,

$$\overrightarrow{B'E}\cdot\overrightarrow{CA}=\frac{1}{2}(PC^2-PA^2),\quad \overrightarrow{C'F}\cdot\overrightarrow{AB}=\frac{1}{2}(PA^2-PB^2),$$

hence (1) holds, by addition.

Conversely, suppose that (1) holds and let P be the point of intersection of the perpendiculars to BC at D and to CA at E. Then, $\overrightarrow{A'P}\cdot\overrightarrow{BC}=\overrightarrow{A'D}\cdot\overrightarrow{BC}$ and $\overrightarrow{B'P}\cdot\overrightarrow{CA}=\overrightarrow{B'E}\cdot\overrightarrow{CA}$ and from (1), we obtain

$$\begin{aligned}\overrightarrow{C'F}\cdot\overrightarrow{AB}&=\overrightarrow{PA'}\cdot\overrightarrow{BC}+\overrightarrow{PB'}\cdot\overrightarrow{CA}=\frac{1}{2}(PC^2-PB^2)+\frac{1}{2}(PA^2-PC^2)\\&=\frac{1}{2}(PA^2-PB^2)=-\overrightarrow{PC'}\cdot\overrightarrow{AB}.\end{aligned}$$

Thus, $\overrightarrow{AB}\cdot\overrightarrow{PF}=0$ and so $PF\perp AB$, as desired.

Also solved by ARKADY ALT, San Jose, CA, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; and TITU ZVONARU, Comăneşti, Romania.

3639. [2011: 234, 237] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c = 3$. Prove that

$$\frac{a^2b}{a+b+1}+\frac{b^2c}{b+c+1}+\frac{c^2a}{c+a+1}\ \le\ 1\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

In his solution to Problem 3549 [2011 : 253], Arkady Alt proved that

$$27(a^2b+b^2c+c^2a+abc)\le 4(a+b+c)^3$$

whenever $a,b,c>0$. When in addition $a+b+c=3$, it follows that

$$\begin{aligned}&(4-a)(4-b)(4-c)-a(4-a)(4-b)-b(4-b)(4-c)-c(4-c)(4-a)\\&\qquad=4(a^2+b^2+c^2)+8(ab+bc+ca)-32-(a^2b+b^2c+c^2a)-abc\\&\qquad\ge 4(a+b+c)^2-32-\frac{4}{27}(a+b+c)^3=0,\end{aligned}$$

so that

$$a(4-a)(4-b)+b(4-b)(4-c)+c(4-c)(4-a)\le(4-a)(4-b)(4-c).$$

Since

$$\frac{a^2b}{a+b+1} = \frac{1}{4}\left[a^2b + abc\left(\frac{a}{4-c}\right)\right],$$
$$\frac{b^2c}{b+c+1} = \frac{1}{4}\left[b^2c + abc\left(\frac{b}{4-a}\right)\right],$$
$$\frac{c^2a}{c+a+1} = \frac{1}{4}\left[c^2a + abc\left(\frac{c}{4-b}\right)\right],$$

the left side of the desired inequality is equal to

$$\begin{aligned}&\frac{1}{4}(a^2b+b^2c+c^2a) + \frac{abc}{4}\left[\frac{a}{4-c}+\frac{b}{4-a}+\frac{c}{4-b}\right]\\ &= 1 - \frac{abc}{4} + \frac{abc}{4(4-a)(4-b)(4-c)}\\ &\qquad\qquad \times\left[a(4-a)(4-b)+b(4-b)(4-c)+c(4-c)(4-a)\right]\\ &\le 1.\end{aligned}$$

Also solved by KEE-WAI LAU, Hong Kong, China; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

3640. [2011: 234, 237] *Proposed by Roy Barbara, Lebanese University, Fanar, Lebanon.*

Consider the function $f(x) = -\sqrt[3]{4x^6+6x^3+3}$.

(a) Find the fixed points of $f(x)$, if any.

(b) Find the periodic points with period 2 of $f(x)$, if any.

(c) Prove that $x=-1$ is the unique real number such that x and $f(x)$ are both integers.

Solution by the proposer.

(a) The equation, $x = f(x)$ for the fixed points is equivalent to

$$0 = 4x^6 + 7x^3 + 3 = (x^3+1)(4x^3+3).$$

Thus, the fixed points are $x=-1$ and $x=-(3/4)^{1/3}$.
(b) Points of period 2 satisfy $x = f(f(x))$, which works out to

$$0 = 64u^4 + 192u^3 + 216u^2 + 109u + 21 = (4u^2+7u+3)(16u^2+20u+7),$$

where $u = x^3$. The roots of the first factor yield the fixed points, while the second factor does not have real roots. Therefore, there are no points of prime period 2.

(c) Let $y = f(x)$, so that

$$-y^3 = 4x^6 + 6x^3 + 3 = (2x^3 + 1)^2 + (2x^3 + 1) + 1.$$

It is known that the only solutions of the diophantine equation $-y^3 = z^2 + z + 1$ are $(y, z) = (-1, -1), (-7, -19), (-7, 18)$, but only the first of these leads to the only integer solution $(x, y) = (-1, -1)$ of the equation.

Alternatively, if we set $w = 1 + 2x^3$, $u = 2w^3 - 1$, $v = -2xyw$, we find that

$$\begin{aligned} u^2 - v^3 - 1 &= (4w^6 - 4w^3 + 1) + 8x^3y^3w^3 - 1 = 4w^3(w^3 - 1 + 2x^3y^3) \\ &= 4w^3(8x^9 + 12x^6 + 6x^3 + 2x^3y^3) = 8x^3w^3(4x^6 + 6x^3 + 3 + y^3) = 0. \end{aligned}$$

If x and y are integers, then u and v are integers that satisfy the Catalan equation $u^2 - v^3 = 1$. The only solutions of this equation are $(u, v) = (\pm 1, 0), (\pm 3, 2)$. Only $(u, v) = (-3, 2)$ gives integer values for both x and y, and we find that x and $f(x)$ are integers if and only if $x = -1$.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; and RICHARD I. HESS, Rancho Palos Verdes, CA, USA.

*The diophantine equation $-y^3 = x^2 + x + 1$ was solved by Nagell in 1921 (*Norsk Mat. Forenings Skrifter, Ser. I, No. 2 and No. 3*), while the equation $u^2 - v^3 = 1$ was solved by Euler in 1738. A recent comprehensive history of the diophantine equation $x^p - y^q = 1$, where p and q are not less than 2, is given in the book,* Catalan's conjecture, *by René Schloof (Springer, 2008).*