THE OLYMPIAD CORNER

No. 302

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Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 octobre 2013**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, d'avoir traduit les problèmes.

OC76. Pour tout entier positif n, soit a_n la plus grande puissance de 2 qui apparaît comme un facteur de $5^n - 3^n$. De plus, soit b_n la plus grande puissance de 2 qui divise n. Montrer que, pour tout n,

$$a_n \leq b_n + 3.$$

OC77. Trouver toutes les fonctions $f : (0, \infty) \to (0, \infty)$ telles que, pour tous les $x, y \in (0, \infty)$, on a

$$f(x)f(y) = f(y)f\left(xf(y)\right) + \frac{1}{xy}\,.$$

OC78. Soit $a_1 = 1, a_2 = 5, a_3 = 14, a_4 = 19, \ldots$ la suite des entiers positifs commençant par 1 et suivi par tous les entiers dont la somme des chiffres est divisible par 5. Montrer que pour tout n, on a

$$a_n \leq 5n\,.$$

OC79. Soit D un point différent des sommets sur le côté BC d'un ΔABC. Soit respectivement I, I_1 et I_2 les centres des cercles inscrits de $\Delta ABC, \Delta ABD$ et ΔADC. Soit E le second point d'intersection des cercles circonscrits de ΔAI_1I et ΔADI_2, et soit F le second point d'intersection des cercles circonscrits de ΔAII_2 et ΔAI_1D. Si $AI_1 = AI_2$, montrer que

$$\frac{EI}{FI} \cdot \frac{ED}{FD} = \frac{{EI_1}^2}{{FI_2}^2}\,.$$

OC80. Soit G un graphe simple avec $3n^2$ sommets ($n \geq 2$), tel que le degré de chaque sommet de G ne soit pas plus grand que $4n$, qu'il existe au moins un sommet de degré un, et qu'entre deux sommets quelconques, il y ait un chemin de longueur ≤ 3. Monter que le nombre minimal de sommets que G puisse avoir est égal à $\dfrac{7n^2 - 3n}{2}$.

. . . .

OC76. For any positive integer n, let a_n be the exponent of the largest power of 2 which occurs as a factor of $5^n - 3^n$. Also, let b_n be the exponent of the largest power of 2 which divides n. Show that

$$a_n \leq b_n + 3$$

for all n.

OC77. Find all functions $f : (0, \infty) \to (0, \infty)$ so that for all $x, y \in (0, \infty)$ we have

$$f(x)f(y) = f(y)f\left(xf(y)\right) + \frac{1}{xy}\,.$$

OC78. Let $a_1 = 1, a_2 = 5, a_3 = 14, a_4 = 19, \ldots$ be the sequence of positive integers starting with 1, followed by all integers with the sum of the digits divisible by 5. Prove that for all n we have

$$a_n \leq 5n\,.$$

OC79. Let D be a point different from the vertices on the side BC of a ΔABC. Let I, I_1 and I_2 be the incenters of the $\Delta ABC, \Delta ABD$ respectively ΔADC. Let E be the second intersection point of the circumcircles of the ΔAI_1I and ΔADI_2, and let F be the second intersection point of the circumcircles of the triangles ΔAII_2 and ΔAI_1D. If $AI_1 = AI_2$, prove that

$$\frac{EI}{FI} \cdot \frac{ED}{FD} = \frac{{EI_1}^2}{{FI_2}^2}\,.$$

OC80. Let G be a simple graph with $3n^2$ vertices ($n \geq 2$), such that the degree of each vertex of G is not greater than $4n$, there exists at least a vertex of degree one, and between any two vertices, there is a path of length ≤ 3. Prove that the minimum number of edges that G might have is equal to $\dfrac{7n^2 - 3n}{2}$.

OLYMPIAD SOLUTIONS

OC16. Given $a_1 \geq 1$ and $a_{k+1} \geq a_k + 1$ for all $k = 1, 2, \ldots, n$, show that

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2 .$$

(*Originally question # 3 from 2010 Singapore Mathematical Olympiad, Senior Section, Round 2.*)

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Felix Boos, University of Kaiserslautern, Kaiserslautern, Germany; Marian Dincă, Bucharest, Romania; Oliver Geupel, Brühl, NRW, Germany; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Cománeşti, Romania. We give the solution of Zvonaru .

Let $s_k = a_1 + a_2 + \cdots + a_k$.

We first prove by induction that

$$a_n a_{n-1} \geq 2s_{n-1} .$$

The case $n = 2$ follows immediately from $a_2 \geq a_1 + 1 \geq 2$, while the inductive step is the following:

$$\begin{aligned} a_{n+1}a_n &\geq (a_{n-1} + 2)a_n = a_{n-1}a_n + 2a_n \\ &\geq 2s_{n-1} + 2a_n = 2s_n \end{aligned}$$

Also, let's observe that equality can be obtained only if $a_{n+1} = a_{n-1} + 2$ for all n, that is if $a_n = n$.

Now, we prove the problem by induction.

For $n = 1$ the problem is equivalent to

$$a_1^3 \geq a_1^2 \Leftrightarrow a_1 \geq 1 .$$

We now prove the inductive step:

$$\begin{aligned} a_1^3 + a_2^3 + \cdots + a_{n+1}^3 &\geq s_n^2 + a_{n+1}^3 \geq s_n^2 + a_{n+1}^2(a_n + 1) \\ &= s_n^2 + a_{n+1}^2 a_n + a_{n+1}^2 \geq s_n^2 + 2s_n a_{n+1} + a_{n+1}^2 = s_{n+1}^2 \end{aligned}$$

Equality only holds if $a_n = n$ for all n.

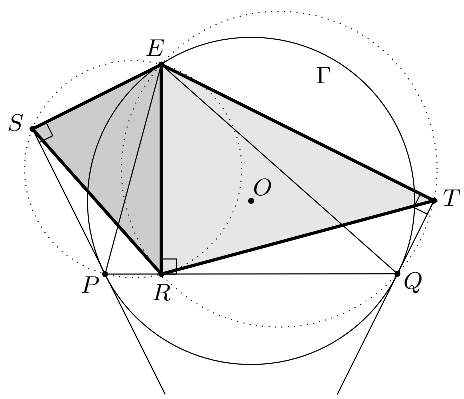
OC17. Prove that the vertices of a convex pentagon $ABCDE$ are concyclic if and only if the following holds

$$d(E, AB) \cdot d(E, CD) = d(E, AC) \cdot d(E, BD) = d(E, AD) \cdot d(E, BC) .$$

(*Originally question # 6 from the 60^{th} national mathematical Olympiad Selection Tests for the Balkan and IMO, 2nd selection test.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

Lemma 1: *Let E, P, Q be distinct points on a circle Γ. Let R, S, T be the feet of the perpendiculars from E onto the line PQ and onto the tangents to Γ in P and Q, respectively. Then, $ER^2 = ES \cdot ET$.*



Proof: Since $\angle ESP + \angle ERP = 180°$ the quadrilateral ESPR is cyclic. Thus

$$\angle ESR = \angle EPQ\,;\ \angle SRE = \angle SPE\,.$$

Since PS is tangent to Γ, we get

$$\angle SPE = \frac{\widehat{PE}}{2} = \angle EQP$$

Thus

$$\angle ESR = \angle EPQ\,;\ \angle SRE = \angle EQP\,.$$

The same argument now works in $ETQR$: $\angle ETQ + \angle ERQ = 180°$, hence the quadrilateral ETQR is cyclic. Thus

$$\angle ETR = \angle EQP\,;\ \angle ERT = \angle EQT = \angle EPQ\,.$$

Thus, $\Delta SRE \sim \Delta RTE$ which implies our claim.

Corollary 2: *If $ABCDE$ is a convex cyclic pentagon, then the following holds:*

$$d(E, AB) \cdot d(E, CD) = d(E, AC) \cdot d(E, BD) = d(E, AD) \cdot d(E, BC). \qquad (1)$$

Proof: Let A_0, B_0, C_0, D_0 be the feet of the perpendiculars from E onto the tangents to the the circumcircle of the pentagon in A, B, C, D, respectively. By Lemma 1, we have $d(E, AB)^2 = EA_0 \cdot EB_0$, $d(E, CD) = EC_0 \cdot ED_0$ and similar relations for the remaining distances. Hence, each expression in the equation (1) equals to $EA_0 \cdot EB_0 \cdot EC_0 \cdot ED_0$.

Lemma 3: *Let $ABCDE$ be a convex pentagon such that the equation (1) holds. Then, the quadrilateral $ABCD$ is cyclic.*

Proof: Let $m = d(E, AB) \cdot d(E, CD)$, $\alpha = \angle AEB$, $\beta = \angle BEC$, $\gamma = \angle CED$. Then, by the relation (1), we have

$$\begin{aligned}
&\frac{m}{AE \cdot BE \cdot CE \cdot DE} \cdot (AC \cdot BD - AB \cdot CD - AD \cdot BC) \\
&= \frac{d(E, AC) \cdot AC}{AE \cdot CE} \cdot \frac{d(E, BD) \cdot BD}{BE \cdot DE} - \frac{d(E, AB) \cdot AB}{AE \cdot BE} \cdot \frac{d(E, CD) \cdot CD}{CE \cdot DE} \\
&\quad - \frac{d(E, AD) \cdot AD}{AE \cdot DE} \cdot \frac{d(E, BC) \cdot BC}{BE \cdot CE} \\
&= \sin(\alpha+\beta)\sin(\beta+\gamma) - \sin\alpha\sin\gamma - \sin(\alpha+\beta+\gamma)\sin\beta \\
&= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\beta\cos\gamma + \cos\beta\sin\gamma) - \sin\alpha\sin\gamma \\
&\quad - (\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma \\
&\qquad\qquad - \sin\alpha\sin\beta\sin\gamma) \cdot \sin\beta \\
&= 0,
\end{aligned}$$

that is, $AC \cdot BD - AB \cdot CD - AD \cdot BC = 0$. By the converse of Ptolemy's Theorem, the points A, B, C, D are concylic.

Corollary 4: *Let $ABCDE$ be a convex pentagon such that the equation (1) holds. Then, the pentagon is cyclic.*

Proof: By Lemma 3, A, B, C, D lie on a common circle Γ. It is enough to show that E also lies on Γ. Let F be the foot of the perpendicular from E onto the line AB. Let Γ be the unit circle in the complex plane and let a, b, c, d, e, f be the coordinates of A, B, C, D, E, F. We have

$$\begin{aligned}
d(E, AB)^2 = (f-e)(\overline{f}-\overline{e}) &= \frac{1}{4}(a+b-e-ab\overline{e})\left(\frac{1}{a}+\frac{1}{b}-\overline{e}-\frac{e}{ab}\right) \\
&= \frac{1}{4ab}(a+b-e-ab\overline{e})^2
\end{aligned}$$

and similar relations for the remaining distances. Direct computation yields

$$16abcd \cdot d(E, AB)^2 \cdot d(E, CD)^2 = \left[e^2 - \sigma e - \tau\overline{e} + (ab+cd)e\overline{e} + (a+b)(c+d)\right]^2,$$

where $\sigma = a+b+c+d$ and $\tau = abc+abd+acd+bcd$, and two analogous identities.

Let z denote the principal value of the square root of the complex number $16abcd \cdot d(E, AB)^2 \cdot d(E, CD)^2$. By (1), there are numbers $\vartheta_1, \vartheta_2, \vartheta_3 \in \{-1,\ 1\}$ such that

$$\begin{aligned}
z &= \vartheta_1\left[e^2 - \sigma e - \tau\overline{e} + (ab+cd)e\overline{e} + (a+b)(c+d)\right] \\
&= \vartheta_2\left[e^2 - \sigma e - \tau\overline{e} + (ac+bd)e\overline{e} + (a+c)(b+d)\right] \\
&= \vartheta_3\left[e^2 - \sigma e - \tau\overline{e} + (ad+bc)e\overline{e} + (a+d)(b+c)\right]
\end{aligned}$$

By the Pigeonhole Principle, two of the numbers ϑ_1, ϑ_2, ϑ_3 must be equal. If, for example, $\vartheta_1 = \vartheta_2$, then we obtain

$$\begin{aligned}
0 &= \vartheta_1\left[(ab+cd)e\overline{e} + (a+b)(c+d) - (ac+bd)e\overline{e} - (a+c)(b+d)\right] \\
&= \vartheta_1(e\overline{e}-1)(a-d)(b-c),
\end{aligned}$$

whence $e\bar{e} = 1$, that is $E \in \Gamma$. The other cases $\vartheta_2 = \vartheta_3$ and $\vartheta_3 = \vartheta_1$ are analogous.

From Corollary 2 and Corollary 4, the equivalence stated in the problem follows.

OC18. Suppose $a_1, a_2, \ldots, a_n$ are n non-zero complex numbers, not necessarily distinct, and k, l are distinct positive integers such that $a_1^k, a_2^k, \ldots, a_n^k$ and $a_1^l, a_2^l, \ldots, a_n^l$ are two identical collections of numbers. Prove that each a_j, $1 \le j \le n$, is a root of unity.
(*Originally question # 2 from the problems used in selection of the Indian team for IMO-2009.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

Let $1 \le j \le n$ be arbitrary. By hypothesis, we can construct inductively a sequence $i_1 = j, i_2, \ldots, i_k, \ldots$ so that for all $m \ge 1$ we have

$$a_{i_m}^k = a_{i_{m+1}}^l .$$

Then, by induction, one can easily prove that

$$a_{i_1}^{k^m} = a_{i_m}^{l^m} .$$

Since $1 \le i_m \le n$ for all m, there exists some $q < r$ such that $i_q = i_r$. Then

$$a_{i_1}^{k^r} = a_{i_r}^{l^r} = a_{i_q}^{l^r} = \left(a_{i_q}^{l^q}\right)^{l^{r-q}} = \left(a_{i_1}^{k^q}\right)^{l^{r-q}} .$$

This implies that

$$a_j^{k^r} = a_j^{k^q l^{r-q}} \Rightarrow a_j^{k^q l^{r-q} - k^r} = 1 .$$

Let's observe that $k^q l^{r-q} - k^r \neq 0$, since otherwise we would have $k^{r-q} = l^{r-q}$, and hence $k = l$. Thus, we proved that there exists some integer $p \neq 0$ so that

$$a_j^p = 1 .$$

This shows that a_j is a root of unity.

OC19. There were 64 contestants at a chess tournament. Every pair played a game that ended either with one of them winning or in a draw. If a game ended in a draw, then each of the remaining 62 contestants won against at least one of these two contestants. There were at least two games ending in a draw at the tournament. Show that we can line up all the contestants so that each of them has won against the one standing right behind him.
(*Originally question # 3 from 53rd national mathematical Olympiad in Slovenia, 3rd selection Exam for the IMO 2009.*)

Solved by Oliver Geupel, Brühl, NRW, Germany.

We prove by induction that the claim holds for any number $n \ge 4$ of contestants.

Let us write $A > B$, $A < B$, or $A \sim B$ if A won against B, A lost against B, or A tied with B, respectively. Let $\mathcal{P}$ denote the property that, if a game ended in a draw, then each of the remaining $n-2$ players won against at least one of these two contestants. Observe that no player tied with more than one other player. For, if A tied with B and C, where $B \neq C$, then by hypothesis $\mathcal{P}$, we have $B > C$ and $C > B$, a contradiction.

We are now ready to prove our initial claim by mathematical induction.

In the base case there are four players A, B, C, and D. Since no contestant had more than one tie, there is no loss of generality in assuming $A \sim B$ and $C \sim D$. By symmetry we may further assume $A > C$. Condition $\mathcal{P}$ now yields $C > B$ and $B > D$. Then,

$$A > C > B > D$$

is a chain with the desired property, which completes the base case.

For the induction step, assume that the claim holds for each tournament of $n = k \geq 4$ contestants. Consider a tournament of $k+1$ players $A_1, \ldots, A_{k+1}$. If each contestant had a game that ended in a draw, then the total number of draws is

$$\left\lceil \frac{k+1}{2} \right\rceil \geq 3,$$

and there are at least two draws in the subtournament of the players $A_1, \ldots, A_k$. Otherwise there is a player, say A_{k+1}, without any draws. In each case there are two draws in the subtournament of the contestants $A_1, \ldots, A_k$. By the induction hypothesis, these k contestants can be arranged in a descending chain, say, $A_1 > \cdots > A_k$. By the property $\mathcal{P}$, the contestant A_{k+1} won against another player. Let i be the least index such that $A_{k+1} > A_i$.

If $i = 1$, then

$$A_{k+1} > A_1 > A_2 > \cdots > A_k$$

is a descending chain of length $k+1$ and we are done.

It remains to consider $i > 1$. If $A_{i-1} \sim A_{k+1}$ then A_i lost against both A_{i-1} and A_{k+1}, which contradicts $\mathcal{P}$. Thus, $A_{i-1} > A_{k+1}$. We obtain the chain

$$A_1 > \cdots > A_{i-1} > A_{k+1} > A_i > \cdots > A_k.$$

This completes the induction.

OC20. Given an integer $n \geq 2$, determine the maximum value the sum $x_1 + x_2 + \cdots + x_n$ may achieve, as the x_i run through the positive integers subject to $x_1 \leq x_2 \leq \cdots \leq x_n$ and $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$.
(*Originally question #10 from 60th national mathematical Olympiad selection tests for the Balkan and IMO, 4th selection test.*)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Cománeşti, Romania. We give the solution of Zvonaru.

We denote

$$s_n = x_1 + x_2 + \cdots + x_n\,.$$

We claim that the maximum value s_n can achieve is $2n$.

If $x_1 = 0$ then $x_2 + \cdots + x_n = 0$, thus $s_n = 0$.

Now, suppose $x_1 \geq 1$. If $x_n = 1$, then $x_1 = ... = x_n = 1$ and hence $n = x_1 + \cdots + x_n = x_1 \cdots x_n = 1$, a contradiction to $n \geq 2$. Hence, for the rest of the proof we can assume

$$1 \leq x_1 \leq x_2 \leq .. \leq x_n \text{ and } x_n \geq 2\,.$$

For $n = 2$, the equality $x_1 + x_2 = x_1x_2$ is equivalent to $(x_1 - 1)(x_2 - 1) = 1$, hence $x_1 = x_2 = 2$ and $s_2 = 4$.

Suppose now that $n \geq 3$. It is easy to see that the equality

$$x_1 + \cdots + x_n = x_1 \cdots x_n$$

can be rewritten as

$$(x_1 - 1)(x_2 - 1) + (x_1x_2 - 1)(x_3 - 1) + \cdots + (x_1x_2 \cdots x_{n-1} - 1)(x_n - 1) = n - 1\,.$$

Thus

$$(x_1x_2 \cdots x_{n-1} - 1)(x_n - 1) \leq n - 1\,. \tag{1}$$

Since $(x_1x_2 \cdots x_{n-1} - 1)x_n = x_1 + \cdots + x_{n-1} \neq 0$,

$$x_n - 1 \leq n - 1 \Rightarrow x_n \leq n\,.$$

Now, substituting $x_1 \cdots x_{n-1} = \dfrac{s_n}{x_n}$ into (1), we get

$$\left(\frac{s_n}{x_n} - 1\right)(x_n - 1) \leq n - 1 \Leftrightarrow s_n\frac{x_n - 1}{x_n} \leq n - 1 + x_n - 1$$
$$\Leftrightarrow s_n \leq \frac{x_n^2 + (n-2)x_n}{x_n - 1}\,.$$

Let's observe that

$$\frac{x_n^2 + (n-2)x_n}{x_n - 1} \leq 2n \Leftrightarrow x_n^2 - (n+2)x_n + 2n \leq 0$$
$$\Leftrightarrow (x_n - 2)(x_n - n) \leq 0\,,$$

which is true since $2 \leq x_n \leq n$. This shows that $s_n \leq 2n$.

Moreover, when $x_n = n$, $x_{n-1} = 2$, $x_{n-2} = \cdots = x_1 = 1$, we have $s_n = 2n$, which shows that $2n$ is the maximum value the sum can achieve.