THE OLYMPIAD CORNER

No. 302

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Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le 1 octobre 2013.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l’anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l’anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

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OC76. Pour tout entier positif $n$, soit $a_n$ la plus grande puissance de 2 qui apparaît comme un facteur de $5^n - 3^n$. De plus, soit $b_n$ la plus grande puissance de 2 qui divise $n$. Montrer que, pour tout $n$,

$$a_n \leq b_n + 3.$$

OC77. Trouver toutes les fonctions $f : (0, \infty) \to (0, \infty)$ telles que, pour tous les $x, y \in (0, \infty)$, on a

$$f(x)f(y) = f(y)f(xf(y)) + \frac{1}{xy}.$$

OC78. Soit $a_1 = 1, a_2 = 5, a_3 = 14, a_4 = 19, \ldots$ la suite des entiers positifs commençant par 1 et suivi par tous les entiers dont la somme des chiffres est divisible par 5. Montrer que pour tout $n$, on a

$$a_n \leq 5n.$$

OC79. Soit $D$ un point différent des sommets sur le côté $BC$ d’un $\triangle ABC$. Soit respectivement $I, I_1$ et $I_2$ les centres des cercles inscrits de $\triangle ABC, \triangle ABD$ et $\triangle ADC$. Soit $E$ le second point d’intersection des cercles circonscrits de $\triangle AI_1I$ et $\triangle ADI_2$, et soit $F$ le second point d’intersection des cercles circonscrits de $\triangle AI_1D$ et $\triangle AI_2D$. Si $AI_1 = AI_2$, montrer que

$$\frac{EI}{FI} \cdot \frac{ED}{FD} = \frac{EI_1^2}{FI_2^2}.$$
**OC80.** Soit $G$ un graphe simple avec $3n^2$ sommets ($n \geq 2$), tel que le degré de chaque sommet de $G$ ne soit pas plus grand que $4n$, qu'il existe au moins un sommet de degré un, et qu'entre deux sommets quelconques, il y ait un chemin de longueur $\leq 3$. Montrer que le nombre minimal de sommets que $G$ puisse avoir est égal à $\frac{7n^2 - 3n^2}{2}$.

**OC76.** For any positive integer $n$, let $a_n$ be the exponent of the largest power of 2 which occurs as a factor of $5^n - 3^n$. Also, let $b_n$ be the exponent of the largest power of 2 which divides $n$. Show that

$$a_n \leq b_n + 3$$

for all $n$.

**OC77.** Find all functions $f : (0, \infty) \to (0, \infty)$ so that for all $x, y \in (0, \infty)$ we have

$$f(x)f(y) = f(y)f(xf(y)) + \frac{1}{xy}.$$

**OC78.** Let $a_1 = 1, a_2 = 5, a_3 = 14, a_4 = 19, \ldots$ be the sequence of positive integers starting with 1, followed by all integers with the sum of the digits divisible by 5. Prove that for all $n$ we have

$$a_n \leq 5n.$$  

**OC79.** Let $D$ be a point different from the vertices on the side $BC$ of a $\triangle ABC$. Let $I, I_1$ and $I_2$ be the incenters of the $\triangle ABC, \triangle ABD$ respectively $\triangle ADC$. Let $E$ be the second intersection point of the circumcircles of the $\triangle AI_1I$ and $\triangle ADI_2$, and let $F$ be the second intersection point of the circumcircles of the triangles $\triangle AI_2$ and $\triangle AI_1D$. If $AI_1 = AI_2$, prove that

$$\frac{EI}{FI} \cdot \frac{ED}{FD} = \frac{EI_1^2}{FI_2^2}.$$

**OC80.** Let $G$ be a simple graph with $3n^2$ vertices ($n \geq 2$), such that the degree of each vertex of $G$ is not greater than $4n$, there exists at least a vertex of degree one, and between any two vertices, there is a path of length $\leq 3$. Prove that the minimum number of edges that $G$ might have is equal to $\frac{7n^2 - 3n^2}{2}$.  

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OC16. Given $a_1 \geq 1$ and $a_{k+1} \geq a_k + 1$ for all $k = 1, 2, \ldots, n$, show that

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2.$$  

(Originally question # 3 from 2010 Singapore Mathematical Olympiad, Senior Section, Round 2.)

Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Felix Boos, University of Kaiserslautern, Kaiserslautern, Germany; Marian Dincă, Bucharest, Romania; Oliver Geupel, Brühl, NRW, Germany; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Comănești, Romania. We give the solution of Zvonaru.

Let $s_k = a_1 + a_2 + \cdots + a_k$.

We first prove by induction that $a_{n+1}a_n \geq 2s_{n-1}$.

The case $n = 2$ follows immediately from $a_2 \geq a_1 + 1 \geq 2$, while the inductive step is the following:

$$a_{n+1}a_n \geq (a_{n-1} + 2)a_n = a_{n-1}a_n + 2a_n \geq 2s_{n-1} + 2a_n = 2s_n$$

Also, let’s observe that equality can be obtained only if $a_{n+1} = a_n - 1 + 2$ for all $n$, that is if $a_n = n$.

Now, we prove the problem by induction.

For $n = 1$ the problem is equivalent to

$$a_1^3 \geq a_1^2 \iff a_1 \geq 1.$$  

We now prove the inductive step:

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq s_n^2 + a_{n+1}^3 \geq s_n^2 + a_{n+1}^2(a_n + 1)$$

$$= s_n^2 + a_{n+1}^2a_n + a_{n+1}^2 \geq s_n^2 + 2s_n a_{n+1} + a_{n+1}^2 = s_{n+1}^2$$

Equality only holds if $a_n = n$ for all $n$.

OC17. Prove that the vertices of a convex pentagon $ABCDE$ are concyclic if and only if the following holds

$$d(E, AB) \cdot d(E, CD) = d(E, AC) \cdot d(E, BD) = d(E, AD) \cdot d(E, BC).$$

(Originally question # 6 from the 60th national mathematical Olympiad Selection Tests for the Balkan and IMO, 2nd selection test.)
Lemma 1: Let $E, P, Q$ be distinct points on a circle $\Gamma$. Let $R, S, T$ be the feet of the perpendiculars from $E$ onto the line $PQ$ and onto the tangents to $\Gamma$ in $P$ and $Q$, respectively. Then, $ER^2 = ES \cdot ET$.

Proof: Since $\angle ESP + \angle ERP = 180^\circ$ the quadrilateral ESPR is cyclic. Thus

$$\angle ESR = \angle EPQ; \angle SRE = \angle SPE.$$ 

Since $PS$ is tangent to $\Gamma$, we get

$$\angle SPE = \frac{\overarc{PE}}{2} = \angle EQP$$

Thus

$$\angle ESR = \angle EPQ; \angle SRE = \angle EQP.$$ 

The same argument now works in ETQR: $\angle ETQ + \angle ERQ = 180^\circ$, hence the quadrilateral ETQR is cyclic. Thus

$$\angle ETR = \angle EQP; \angle ERT = \angle EQT = \angle EPQ.$$ 

Thus, $\triangle SRE \sim \triangle RTE$ which implies our claim.

Corollary 2: If $ABCDE$ is a convex cyclic pentagon, then the following holds:

$$d(E, AB) \cdot d(E, CD) = d(E, AC) \cdot d(E, BD) = d(E, AD) \cdot d(E, BC). \quad (1)$$

Proof: Let $A_0, B_0, C_0, D_0$ be the feet of the perpendiculars from $E$ onto the tangents to the the circumcircle of the pentagon in $A, B, C, D$, respectively. By Lemma 1, we have $d(E, AB)^2 = EA_0 \cdot EB_0$, $d(E, CD) = EC_0 \cdot ED_0$ and similar relations for the remaining distances. Hence, each expression in the equation (1) equals to $EA_0 \cdot EB_0 \cdot EC_0 \cdot ED_0$.

Lemma 3: Let $ABCDE$ be a convex pentagon such that the equation (1) holds. Then, the quadrilateral $ABCD$ is cyclic.
If, for example, \( \vartheta = \angle AEB \), then, by the relation (1), we have
\[
\begin{align*}
\frac{m}{\vartheta} &= \frac{d(E, AB) \cdot d(E, CD), \alpha = \angle AEB, \beta = \angle BEC, \gamma = \angle CDE.}
\end{align*}
\]

**Proof:** Let \( m = d(E, AB) \cdot d(E, CD) \). Then, by the relation (1), we have
\[
\begin{align*}
& \frac{m}{\vartheta} \left( \frac{AC \cdot BD - AB \cdot CD - AD \cdot BC}{AE \cdot CE \cdot DE} \right) \\
& = \frac{\frac{d(E, AC)}{\vartheta} \cdot AC \cdot \frac{d(E, BD)}{\vartheta} \cdot BD - \frac{d(E, AB)}{\vartheta} \cdot \frac{d(E, CD)}{\vartheta} \cdot CD}{\frac{AE}{\vartheta} \cdot \frac{BE}{\vartheta} \cdot \frac{CE}{\vartheta} \cdot \frac{DE}{\vartheta}} \\
& = \frac{1}{\vartheta} \left( \frac{d(E, AC) \cdot AC}{AE} \cdot \frac{d(E, BD) \cdot BD}{BE} - \frac{d(E, AB) \cdot AB}{AE} \cdot \frac{d(E, CD) \cdot CD}{CE} \right) \\
& = \frac{1}{\vartheta} \left( \frac{d(E, AC) \cdot AC}{AE} \cdot \frac{d(E, BD) \cdot BD}{BE} - \frac{d(E, AB) \cdot AB}{AE} \cdot \frac{d(E, CD) \cdot CD}{CE} \right) \\
& = \sin(\alpha + \beta) \sin(\beta + \gamma) - \sin \alpha \sin \gamma - \sin(\alpha + \beta + \gamma) \sin \beta \\
& = (\sin \alpha \cos \beta + \cos \alpha \sin \beta)(\sin \beta \cos \gamma + \cos \beta \sin \gamma) - \sin \alpha \sin \gamma \\
& - (\sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma) \\
& - \sin \alpha \sin \beta \sin \gamma \cdot \sin \beta \\
& = 0,
\end{align*}
\]

that is, \( AC \cdot BD - AB \cdot CD - AD \cdot BC = 0 \). By the converse of Ptolemy’s Theorem, the points \( A \), \( B \), \( C \), \( D \) are concyclic.

**Corollary 4:** Let \( ABCDE \) be a convex pentagon such that the equation (1) holds. Then, the pentagon is cyclic.

**Proof:** By Lemma 3, \( A \), \( B \), \( C \), \( D \) lie on a common circle \( \Gamma \). It is enough to show that \( E \) also lies on \( \Gamma \). Let \( F \) be the foot of the perpendicular from \( E \) onto the line \( AB \). Let \( \Gamma \) be the unit circle in the complex plane and let \( a, b, c, d, e, f \) be the coordinates of \( A, B, C, D, E, F \). We have
\[
d(E, AB)^2 = (f - e)(\overline{f} - \overline{e}) = \frac{1}{4} \left( a + b - e - ab\overline{e} \right) \left( \frac{1}{a} + \frac{1}{b} - \overline{e} - \overline{e} \overline{a} \right)
\]
\[
= \frac{1}{4ab} \left( a + b - e - ab\overline{e} \right) \left( a + b - e - ab\overline{e} \right)
\]
and similar relations for the remaining distances. Direct computation yields
\[
16abcd \cdot d(E, AB)^2 \cdot d(E, CD)^2 = \left[ e^2 - \sigma e - \tau \overline{e} + (ab + cd)e\overline{e} + (a + b)(c + d) \right]^2,
\]
where \( \sigma = a + b + c + d \) and \( \tau = abc + abd + acd + bcd \), and two analogous identities.

Let \( z \) denote the principal value of the square root of the complex number \( 16abcd \cdot d(E, AB)^2 \cdot d(E, CD)^2 \). By (1), there are numbers \( \vartheta_1, \vartheta_2, \vartheta_3 \in \{-1, 1\} \) such that
\[
z = \vartheta_1 \left[ e^2 - \sigma e - \tau \overline{e} + (ab + cd)e\overline{e} + (a + b)(c + d) \right]
\]
\[
= \vartheta_2 \left[ e^2 - \sigma e - \tau \overline{e} + (ac + bd)e\overline{e} + (a + c)(b + d) \right]
\]
\[
= \vartheta_3 \left[ e^2 - \sigma e - \tau \overline{e} + (ad + bc)e\overline{e} + (a + d)(b + c) \right]
\]

By the Pigeonhole Principle, two of the numbers \( \vartheta_1, \vartheta_2, \vartheta_3 \) must be equal. If, for example, \( \vartheta_1 = \vartheta_2 \), then we obtain
\[
0 = \vartheta_1 \left[ (ab + cd)e\overline{e} + (a + b)(c + d) - (ac + bd)e\overline{e} - (a + c)(b + d) \right]
\]
\[
= \vartheta_1(e\overline{e} - 1)(a - d)(b - c),
\]
whence $e^{\theta_1} = 1$, that is $E \in \Gamma$. The other cases $\theta_2 = \theta_3$ and $\theta_3 = \theta_1$ are analogous.

From Corollary 2 and Corollary 4, the equivalence stated in the problem follows.

**OC18.** Suppose $a_1, a_2, \ldots, a_n$ are $n$ non-zero complex numbers, not necessarily distinct, and $k$, $l$ are distinct positive integers such that $a_1^k, a_2^k, \ldots, a_n^k$ and $a_1^l, a_2^l, \ldots, a_n^l$ are two identical collections of numbers. Prove that each $a_j$, $1 \leq j \leq n$, is a root of unity.

*(Originally question # 2 from the problems used in selection of the Indian team for IMO-2009.)*

*Solved by Oliver Geupel, Brühl, NRW, Germany.*

Let $1 \leq j \leq n$ be arbitrary. By hypothesis, we can construct inductively a sequence $i_1 = j, i_2, \ldots, i_k, \ldots$ so that for all $m \geq 1$ we have

$$a_{i_m}^k = a_{i_{m+1}}^l.$$

Then, by induction, one can easily prove that

$$a_{i_1}^k = a_{i_m}^m.$$

Since $1 \leq i_m \leq n$ for all $m$, there exists some $q < r$ such that $i_q = i_r$. Then

$$a_{i_1}^{k^r} = a_{i_r}^r = a_{i_q}^r = \left(a_{i_q}^q\right)^{l^r-q} = \left(a_{i_1}^q\right)^{l^r-q}.$$

This implies that

$$a_{i_1}^{k^r} = a_{i_1}^{l^r-q} \Rightarrow a_{i_1}^{k^r-(l^r-q)} = 1.$$

Let’s observe that $k^{l^r-q} - k^r \neq 0$, since otherwise we would have $l^r-q = l^r-q$, and hence $k = l$. Thus, we proved that there exists some integer $p \neq 0$ so that

$$a_{i_1}^p = 1.$$

This shows that $a_j$ is a root of unity.

**OC19.** There were 64 contestants at a chess tournament. Every pair played a game that ended either with one of them winning or in a draw. If a game ended in a draw, then each of the remaining 62 contestants won against at least one of these two contestants. There were at least two games ending in a draw at the tournament. Show that we can line up all the contestants so that each of them has won against the one standing right behind him.

*(Originally question # 3 from 53rd national mathematical Olympiad in Slovenia, 3rd selection Exam for the IMO 2009.)*

*Solved by Oliver Geupel, Brühl, NRW, Germany.*

We prove by induction that the claim holds for any number $n \geq 4$ of contestants.
Let us write $A > B$, $A < B$, or $A \sim B$ if $A$ won against $B$, $A$ lost against $B$, or $A$ tied with $B$, respectively. Let $\mathcal{P}$ denote the property that, if a game ended in a draw, then each of the remaining $n - 2$ players won against at least one of these two contestants. Observe that no player tied with more than one other player. For, if $A$ tied with $B$ and $C$, where $B \neq C$, then by hypothesis $\mathcal{P}$, we have $B > C$ and $C > B$, a contradiction.

We are now ready to prove our initial claim by mathematical induction.

In the base case there are four players $A, B, C,$ and $D$. Since no contestant had more than one tie, there is no loss of generality in assuming $A \sim B$ and $C \sim D$. By symmetry we may further assume $A > C$. Condition $\mathcal{P}$ now yields $C > B$ and $B > D$. Then,

$$A > C > B > D$$

is a chain with the desired property, which completes the base case.

For the induction step, assume that the claim holds for each tournament of $n = k \geq 4$ contestants. Consider a tournament of $k + 1$ players $A_1, \ldots, A_{k+1}$. If each contestant had a game that ended in a draw, then the total number of draws is

$$\left\lceil \frac{k+1}{2} \right\rceil \geq 3,$$

and there are at least two draws in the subtournament of the players $A_1, \ldots, A_k$. Otherwise there is a player, say $A_{k+1}$, without any draws. In each case there are two draws in the subtournament of the contestants $A_1, \ldots, A_k$. By the induction hypothesis, these $k$ contestants can be arranged in a descending chain, say, $A_1 > \cdots > A_k$. By the property $\mathcal{P}$, the contestant $A_{k+1}$ won against another player. Let $i$ be the least index such that $A_{k+1} > A_i$.

If $i = 1$, then

$$A_{k+1} > A_1 > A_2 > \cdots > A_k$$

is a descending chain of length $k + 1$ and we are done.

It remains to consider $i > 1$. If $A_{i-1} \sim A_{k+1}$ then $A_i$ lost against both $A_{i-1}$ and $A_{k+1}$, which contradicts $\mathcal{P}$. Thus, $A_{i-1} > A_{k+1}$. We obtain the chain

$$A_i > \cdots > A_{i-1} > A_{k+1} > A_{k+1} > A_i > \cdots > A_k.$$ 

This completes the induction.

**OC20.** Given an integer $n \geq 2$, determine the maximum value the sum $x_1 + x_2 + \cdots + x_n$ may achieve, as the $x_i$ run through the positive integers subject to $x_1 \leq x_2 \leq \cdots \leq x_n$ and $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$.

*(Originally question #10 from 60th national mathematical Olympiad selection tests for the Balkan and IMO, 4th selection test.)*

_Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru._

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We denote 
\[ s_n = x_1 + x_2 + \cdots + x_n. \]

We claim that the maximum value \( s_n \) can achieve is \( 2n \).

If \( x_1 = 0 \) then \( x_2 + \cdots + x_n = 0 \), thus \( s_n = 0 \).

Now, suppose \( x_1 \geq 1 \). If \( x_n = 1 \), then \( x_1 = \cdots = x_n = 1 \) and hence
\[ n = x_1 + \cdots + x_n = x_1 \cdots x_n = 1, \]
a contradiction to \( n \geq 2 \). Hence, for the rest of the proof we can assume

\[ 1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \text{ and } x_n \geq 2. \]

For \( n = 2 \), the equality \( x_1 + x_2 = x_1x_2 \) is equivalent to \( (x_1 - 1)(x_2 - 1) = 1 \), hence \( x_1 = x_2 = 2 \) and \( s_2 = 4 \).

Suppose now that \( n \geq 3 \). It is easy to see that the equality
\[ x_1 + \cdots + x_n = x_1 \cdots x_n \]
can be rewritten as
\[ (x_1 - 1)(x_2 - 1) + (x_1x_2 - 1)(x_3 - 1) + \cdots + (x_1x_2 \cdots x_{n-1} - 1)(x_n - 1) = n - 1. \]

Thus
\[ (x_1x_2 \cdots x_{n-1} - 1)(x_n - 1) \leq n - 1. \quad (1) \]

Since \( (x_1x_2 \cdots x_{n-1} - 1)x_n = x_1 + \cdots + x_{n-1} \neq 0 \),
\[ x_n - 1 \leq n - 1 \Rightarrow x_n \leq n. \]

Now, substituting \( x_1 \cdots x_{n-1} = \frac{s_n}{x_n} \) into (1), we get
\[ \left( \frac{s_n}{x_n} - 1 \right)(x_n - 1) \leq n - 1 \Leftrightarrow s_n \frac{x_n - 1}{x_n} \leq n - 1 + x_n - 1 \]
\[ \Leftrightarrow s_n \leq \frac{x_n^2 + (n-2)x_n}{x_n - 1}. \]

Let’s observe that
\[ \frac{x_n^2 + (n-2)x_n}{x_n - 1} \leq 2n \Leftrightarrow x_n^2 - (n+2)x_n + 2n \leq 0 \]
\[ \Leftrightarrow (x_n - 2)(x_n - n) \leq 0, \]
which is true since \( 2 \leq x_n \leq n \). This shows that \( s_n \leq 2n \).

Moreover, when \( x_n = n, x_{n-1} = 2, x_{n-2} = \cdots = x_1 = 1 \), we have \( s_n = 2n \),
which shows that \( 2n \) is the maximum value the sum can achieve.