SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.


Let a tetrahedron $ABCD$ with centroid $G$ be inscribed in a sphere of radius $R$. The lines $AG, BG, CG, DG$ meet the sphere again at $A_1, B_1, C_1, D_1$ respectively. The edges of the tetrahedron are denoted $a, b, c, d, e, f$. Prove or disprove that

$$\frac{4}{R} \leq \frac{1}{GA_1} + \frac{1}{GB_1} + \frac{1}{GC_1} + \frac{1}{GD_1} \leq \frac{4\sqrt{6}}{9} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} \right).$$

II. Solution by Tomasz Cieślą, student, University of Warsaw, Poland.

Murray Klamkin proved a generalization of the right-hand inequality [1996:183-184] for an $n$-dimensional simplex. He then conjectured that a generalization of the left-hand inequality likewise held: Let $G$ and $O$ be the centroid and circum-centre of an $n$-dimensional simplex $A_0A_1...A_n$ inscribed in a sphere of radius $R$. Let each line $A_iG$ meet the sphere again in $A'_i$ ($i = 0, \ldots, n$). Then

$$\frac{n+1}{R} \leq \sum \frac{1}{A'_iG}.$$ 

If $P$ and $Q$ are the endpoints of the diameter through $G$, then $A_iG \cdot A'_iG = PG \cdot QG = (R+OG)(R-OG) = R^2-OG^2$, whence our inequality is equivalent to $R \sum A_iG \geq (n+1)(R^2-OG^2)$. Moreover, because $(n+1)(R^2-OG^2) = \sum A_iG^2$ (a known equality that can be proved, for example, by a vector argument with the origin at $O$), the problem is reduced to proving that

$$R \sum A_iG \geq \sum A_iG^2.$$ 

Surprisingly, we can prove it using only the triangle inequality! Looking at triangle $A_iOG$ we can write $|OA_i - A_iG| \leq OG$. After squaring we get $OA_i^2 + A_iG^2 - 2OA_i \cdot A_iG \leq OG^2$, which is equivalent to $2R \cdot A_iG \geq A_iG^2 + R^2 - OG^2$. Summing up $n+1$ such inequalities, we get

$$2R \sum A_iG \geq (n+1)(R^2-OG^2) + \sum A_iG^2 = 2 \sum A_iG^2.$$ 

Done!

Equality holds if and only if points $A_i, O, G$ are collinear for all $i$. This happens when $O = G$, or all points $A_i$ lie on the line $OG$. Clearly, in the latter case, the simplex is degenerate, and every vertex coincides with $X$ or $Y$, where $XY$ is a diameter of the sphere. In the three-dimensional case, $O = G$ implies that the tetrahedron is isosceles; see, for example, the recent solution to problem 478 [2012 : 68-70].

No other solutions have been received.

Crux Mathematicorum, Vol. 38(3), March 2012
Let $a$, $b$, and $c$ be nonnegative real numbers with $a + b + c = 1$. Prove that

$$\frac{27}{128}[(a - b)^2 + (b - c)^2 + (c - a)^2] + \frac{4}{1 + a} + \frac{4}{1 + b} + \frac{4}{1 + c} \leq \frac{3}{ab + bc + ca}.$$ 

Solution by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina, expanded by the editor.

Let $s = ab + bc + ca$ and $t = abc$. Since

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 2 \sum \text{cyclic} a^2 - 2 \sum \text{cyclic} ab$$

$$= 2 \left( \left( \sum \text{cyclic} a \right)^2 - 3 \sum \text{cyclic} ab \right) = 2(1 - 3s)$$

and

$$\frac{4}{1 + a} + \frac{4}{1 + b} + \frac{4}{1 + c} = \frac{4 \sum \text{cyclic} (1 + b)(1 + c)}{(1 + a)(1 + b)(1 + c)}$$

$$= \frac{4 \left( 3 + 2 \sum \text{cyclic} a + \sum \text{cyclic} ab \right)}{1 + \sum \text{cyclic} a + \sum \text{cyclic} ab + abc} = \frac{4(5 + s)}{2 + s + t}$$

the given inequality is equivalent, in succession, to

$$\frac{27(1 - 3s)}{64} + \frac{4(5 + s)}{2 + s + t} \leq \frac{3}{s}$$

$$\frac{27(1 - 3s)}{64} + \left( \frac{4(5 + s)}{2 + s + t} - 9 \right) \leq \frac{3}{s} - 9 = \frac{3 - 9s}{s}$$

$$\frac{2 - 5s - 9t}{2 + s + t} \leq \frac{3(1 - 3s)(64 - 9s)}{64s}$$

$$\frac{-1 + 4s - 9t}{2 + s + t} \leq \frac{3(1 - 3s)(64 - 9s)(2 + s + t) - 64s}{64s(2 + s + t)}$$

Now we apply Schur’s Inequality which states that

$$\sum \text{cyclic} a^3(a - b)(a - c) \geq 0$$

for any $\lambda \geq 0$. Setting $\lambda = 1$ we then have

$$\sum \text{cyclic} a^3 - \sum \text{cyclic} ab(a + b) + 3abc \geq 0.$$ 

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Note that
\[ \sum_{\text{cyclic}} a^3 = \left( \sum_{\text{cyclic}} a \right)^3 - 3 \sum_{\text{cyclic}} ab(a + b) - 3abc \] (3)
and
\[ \sum_{\text{cyclic}} ab(a + b) = \left( \sum_{\text{cyclic}} a \right) \left( \sum_{\text{cyclic}} ab \right) - 3abc. \] (4)
Substituting (3) and (4) into (2) we obtain
\[ \left( \sum_{\text{cyclic}} a \right)^3 - 4 \left( \sum_{\text{cyclic}} a \right) \left( \sum_{\text{cyclic}} ab \right) + 9abc \geq 0 \]
so \( 1 + 9t \geq 4s \) or \(-1 + 4s - 9t \leq 0\). On the other hand,
\[ 1 - 3s = \left( \sum_{\text{cyclic}} a \right)^2 - 3 \sum_{\text{cyclic}} ab = \sum_{\text{cyclic}} a^2 - \sum_{\text{cyclic}} ab = \frac{1}{2} \sum_{\text{cyclic}} (a - b)^2 \geq 0 \]
so \(3s \leq 1\) and
\[ (64 - 9s)(2 + s + t) \geq (64 - 3)(2) = 122 \geq (122)(3s) \geq 64s. \]
Hence (1) holds and the proof is complete. If equality holds in (1) then we must have \(s = \frac{1}{3}\) and \(4s - 9t - 1 = 0\) so \(t = \frac{1}{27}\). But \(\frac{1}{27} = t = abc \leq \left( \frac{a + b + c}{3} \right)^3 = \frac{1}{27}\) so equality holds which implies \(a = b = c = \frac{1}{3}\).

Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIČ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy(two proofs); and the proposer.

Lau pointed out that in order for the inequality to make sense, we must stipulate that at most one of \(a\), \(b\), and \(c\) is zero. Perfetti proved the stronger result in which \(\frac{27}{128}\) is replaced by \(\frac{1}{4}\).


Let \(ABCD\) be a quadrilateral.

(a) Find sufficient and necessary conditions on the sides and angles of \(ABCD\), so that there is an inner point \(P\) such that two perpendicular lines through \(P\) divide the quadrilateral \(ABCD\) into four quadrilaterals of equal area.

(b) Determine \(P\).

The problem remains open. We received comments from Václav Konečný, Big Rapids, MI, USA and from Peter Y. Woo, Biola University, La Mirada, CA,
USA. There is a simple topological argument (which applies to any bounded region of the Euclidean plane) proving the existence of two perpendicular lines that divide the area into four regions of equal area, but that argument fails to address our problem’s most interesting aspects. Is the choice of perpendicular lines unique for a quadrilateral? No for a square, yes for any other parallelogram, and still open (as far as we know) for the general quadrilateral. Are the four equal-area regions that result all quadrilaterals? No for a nonsquare rhombus (the regions are all triangles), yes for other parallelograms, while the answer varies for other quadrilaterals; for example, if the length of one of its sides is close to zero (so that the quadrilateral is nearly a triangle), then one of the four equal-area regions might be a triangle and one a pentagon. As for part (b), even for the simple case of an isosceles trapezoid, the location of the point \( P \) in terms of the sides and angles might be too complicated to be of any interest.


Let \( z_1, z_2, z_3, z_4 \) be distinct complex numbers with the same modulus, \( \alpha = |(z_3 - z_2)(z_3 - z_4)|, \beta = |(z_1 - z_2)(z_1 - z_4)| \) and

\[
u(\epsilon) = \frac{\alpha(z_1 - z_4) + \epsilon \beta(z_3 - z_4)}{\alpha(z_1 - z_2) + \epsilon \beta(z_3 - z_2)}.
\]

Prove that \( u(+1) \) or \( u(-1) \) is a real number.

**Solution by the proposer.**

We take \( z_1, z_2, z_3, z_4 \) to represent the points \( M_1, M_2, M_3, M_4 \) on a circle \( \Gamma \) centered at the origin of the Argand Plane. Note that \( \alpha \) and \( \beta \) are positive real numbers and that the four numbers \( \alpha(z_1 - z_4) \pm \beta(z_3 - z_4), \alpha(z_1 - z_2) \pm \beta(z_3 - z_2) \) are nonzero (since the vectors \( M_1M_4 \) and \( M_3M_2 \) are linearly independent, as are \( M_1M_2 \) and \( M_3M_4 \)). We distinguish two cases according as lines \( M_1M_3 \) and \( M_2M_4 \) are parallel or not.

(a) If \( M_1M_3 \parallel M_2M_4 \), then \( M_1, M_2, M_3, M_4 \) are the vertices of an isosceles trapezoid inscribed in \( \Gamma \), so that \( M_3M_4 = M_1M_2 \) and \( M_3M_2 = M_1M_4 \). Thus,

\[
\alpha = |z_3 - z_2||z_3 - z_4| = M_3M_2 \cdot M_3M_4 = M_1M_4 \cdot M_1M_2 = |z_2 - z_1||z_4 - z_1| = \beta.
\]

and then

\[
u(-1) = \frac{z_1 - z_4 - (z_3 - z_4)}{z_1 - z_2 - (z_3 - z_2)} = \frac{z_1 - z_3}{z_1 - z_3} = 1.
\]

(b) Suppose that lines \( M_1M_3 \) and \( M_2M_4 \) intersect at a point \( M \) that is represented by the complex number \( z \). Note that because the \( z_i \) are distinct, none of them can equal \( z \); the real numbers \( \lambda \) and \( \mu \) defined by \( z - z_1 = \lambda(z - z_3) \) and \( z - z_4 = \mu(z - z_2) \) are therefore nonzero. Because \( \lambda \neq 1 \), the first equation says that

\[
z = \frac{1}{1 - \lambda}(z_1 - \lambda z_3);
\]

plugging that value of \( z \) into the second equation gives us

\[
\frac{z_1 - z_4 - \lambda(z_3 - z_4)}{z_1 - z_2 - \lambda(z_3 - z_2)} = \mu.
\]
It therefore suffices to prove that $|\lambda| = \frac{\beta}{\alpha}$. Because the $M_i$ are concyclic, we deduce that $\Delta M_3MM_4$ and $\Delta M_2MM_1$ are inversely similar, as are $\Delta M_1MM_4$ and $\Delta M_2MM_3$. Consequently, 

$$\frac{M_2M_1}{M_3M_4} = \frac{MM_1}{MM_4} = \frac{MM_2}{MM_3}$$

and 

$$\frac{M_1M_4}{M_2M_3} = \frac{MM_1}{MM_4} = \frac{MM_2}{MM_3} = \left(\frac{MM_1}{MM_3}\right)^2;$$

As a result we have 

$$\left(\frac{M_2M_1}{M_3M_4} \cdot \frac{M_1M_4}{M_2M_3}\right)^2 = \frac{MM_1}{MM_4} \cdot \frac{MM_2}{MM_3} \cdot \frac{MM_1}{MM_2} \cdot \frac{MM_4}{MM_3} = \left(\frac{MM_1}{MM_3}\right)^2;$$

that is, $\left(\frac{\beta}{\alpha}\right)^2 = \lambda^2$, and the desired equality $|\lambda| = \frac{\beta}{\alpha}$ follows.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany.


Calculate the sum 

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n}\right).$$

I. Solution based on the approach of AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Let $a_0 = 0$, and, for $n \geq 1$, let 

$$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n},$$

$$S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} a_k = \sum_{k=1}^{n} (a_k - a_{k-1}) a_k,$$

and 

$$T_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} a_{k-1} = \sum_{k=1}^{n} (a_k - a_{k-1}) a_{k-1}.$$ 

Then 

$$S_n + T_n = \sum_{k=1}^{n} (a_k - a_{k-1})(a_k + a_{k-1}) = \sum_{k=1}^{n} (a_k^2 - a_{k-1}^2) = a_n^2,$$

and 

$$S_n - T_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} (a_k - a_{k-1}) = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Therefore $S_n = \frac{1}{2} \left[ a_n^2 + \sum_{k=1}^{n} \frac{1}{k^2} \right]$. The desired sum is equal to 

$$\lim_{n \to \infty} S_n = \frac{1}{2} \left[ (\log 2)^2 + \frac{\pi^2}{6} \right].$$

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II. Solution following approach of Richard I. Hess, Rancho Palos Verdes, CA, USA; Kee-Wai Lau, Hong Kong, China; the Missouri State University Problem Solving Group, Springfield, MO; and the proposer.

For positive integer $m$, let

$$S_m = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = \sum_{1 \leq k \leq n \leq m} \frac{(-1)^{n-1}}{n} \frac{(-1)^{k-1}}{k}.$$

Interchanging the order of summation and relabeling the indices yields

$$S_m = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{n=k}^{m} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=n}^{m} \frac{(-1)^{k-1}}{k}$$

$$= \sum_{n=1}^{m} \frac{1}{n^2} + \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=n+1}^{m} \frac{(-1)^{k-1}}{k}.$$

Adding the two expressions for $S_m$ yields that

$$S_m = \frac{1}{2} \left[ \sum_{n=1}^{m} \frac{1}{n^2} + \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \right].$$

The required sum is

$$\lim_{m \to \infty} S_m = \frac{\pi^2}{12} + \frac{(\log 2)^2}{2}.$$

III. Solution by Oliver Geupel, Brühl, NRW, Germany (abridged).

When $a_n = \sum_{k=1}^{n} (-1)^{k-1} / k, b_n = \sum_{k=1}^{n} k^{-2}$ and $c_n = \sum_{k=1}^{n} (-1)^{k-1} a_k / k$, it can be proved by induction that

$$c_n = \frac{1}{2} \left( a_n^2 + b_n \right).$$

The required sum is equal to

$$\lim_{n \to \infty} c_n = \frac{1}{2} (\log 2)^2 + \frac{1}{12} \pi^2.$$

IV. Solution based on those of Anastasios Kotrononis, Athens, Greece; Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and Albert Stadler, Herrliberg, Switzerland.

Since

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} = \int_{0}^{1} \sum_{k=1}^{n} (-x)^{k-1} dx = \int_{0}^{1} \frac{1 - (-x)^n}{1 + x} dx,$$

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the proposed sum is equal to
\[
\sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 \frac{1}{1+x} \, dx - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 \frac{(-x)^n}{1+x} \, dx \\
= (\log 2)^2 - \int_0^1 \frac{1}{1+x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-x)^n}{n} \, dx \\
= (\log 2)^2 - \int_0^1 \frac{\log(1-x)}{1+x} \, dx \\
= (\log 2)^2 - \left[ \frac{(\log 2)^2}{2} - \frac{\pi^2}{12} \right] = \frac{(\log 2)^2}{2} + \frac{\pi^2}{12}
\]

No other solutions were received.
Perfetti provided a justification for the interchange of summation and integration in (IV), while Kotronis gave this determination of the final integral:
\[
- \int_0^1 \frac{\log(1-t)}{1+t} \, dt = \int_0^1 \int_{-1}^1 \frac{1}{1+t} \cdot \frac{x}{1+y} \, dx \, dy = \int_{-1}^1 \int_0^1 \frac{t}{(1+t)(1+y)} \, dt \, dy \\
= \int_{-1}^1 \int_0^1 \left[ \frac{1}{(y-1)(1+t)} - \frac{1}{(y-1)(1+y)} \right] \, dt \, dy \\
= \int_{-1}^1 \left[ \frac{\log 2}{y-1} - \frac{\log(1+y)}{y(y-1)} \right] \, dy = -(\log 2)^2 + \int_{-1}^1 \left[ \frac{\log(1+y)}{y} - \frac{\log(1+y)}{y-1} \right] \, dy \\
= -(\log 2)^2 - \int_0^1 \frac{\log(1-x)}{x} \, dx + \int_0^1 \frac{\log(1-x)}{1+x} \, dx,
\]
so that
\[
\int_0^1 \frac{\log(1-t)}{1+t} \, dt = \frac{(\log 2)^2}{2} + \frac{1}{2} \int_0^1 \frac{\log(1-x)}{x} \, dx = \frac{(\log 2)^2}{2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \, dx \\
= \frac{(\log 2)^2}{2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{(\log 2)^2}{2} - \frac{\pi^2}{12}
\]


Let a, b, and c be positive real numbers. Prove that
\[
\sqrt{\frac{a}{a+b}} + \sqrt{\frac{b}{b+c}} + \sqrt{\frac{c}{c+a}} \leq 2 \sqrt{1 + \frac{abc}{(a+b)(b+c)(c+a)}}.
\]

Solution by George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Oliver Geupel, Brühl,

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NRW, Germany; Titu Zvonaru, Comăneşti, Romania; AN-anduud Problem
Solving Group, Ulaanbaatar, Mongolia; and the proposer (independently).

Applying the Cauchy-Schwarz Inequality, we obtain that
\[
\left(\sqrt{\frac{a}{a+b}} + \sqrt{\frac{b}{b+c}} + \sqrt{\frac{c}{c+a}}\right)^2
\leq (a(b+c) + b(c+a) + c(a+b))
\times \left(\frac{1}{(a+b)(b+c)} + \frac{1}{(b+c)(c+a)} + \frac{1}{(c+a)(a+b)}\right)
= 2(ab + bc + ca) \left(\frac{2(a+b+c)}{(a+b)(b+c)(c+a)}\right)
= 4 \frac{(a+b)(b+c)(c+a) + abc}{(a+b)(b+c)(c+a)} = 4 \left(1 + \frac{abc}{(a+b)(b+c)(c+a)}\right).
\]
from which the desired result follows. Equality holds if and only if
\[
a(b+c) = b(c+a) = c(a+b) = \frac{abc}{(a+b)(b+c)(c+a)} \Rightarrow abc = \frac{a^2b}{b+a} = \frac{b^2c}{c+b} = \frac{c^2a}{a+c},
\]
deleted text.

Also solved by MICHEL BATAILLE, Rouen, France; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and ALBERT STADLER, Herrliberg, Switzerland.

These solvers also used the Cauchy-Schwarz Inequality. The proposer attributed his solution to Zhao Bin, Ningbo, China. Geupel noted that the proposer posed the same problem to the Art of Problem Solving in 2006 (http://www.artofproblemsolving.com/Forum/viewtopic.php?t=78720).


Let \(x, y, z\) be positive real numbers such that \(x^2 + y^2 + z^2 = 3\). Prove that
\[
\frac{1 + x^2}{z + 2} + \frac{1 + y^2}{x + 2} + \frac{1 + z^2}{y + 2} \geq 2.
\]

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Let \(f(x, y, z)\) denote the left side of the given inequality. By the AM-GM Inequality we have
\[
\frac{1 + x^2}{z + 2} + \frac{1}{9}(1 + x^2)(z + 2) \geq \frac{2}{3}(1 + x^2)
\]
so
\[
\frac{1 + x^2}{z + 2} \geq \frac{4}{9}(1 + x^2) - \frac{1}{9}z(1 + x^2). \tag{1}
\]
Similarly,
\[
\frac{1 + y^2}{x + 2} \geq \frac{4}{9}(1 + y^2) - \frac{1}{9}x(1 + y^2) \tag{2}
\]
and
\[
\frac{1 + z^2}{y + 2} \geq \frac{4}{9}(1 + z^2) - \frac{1}{9}y(1 + z^2). \tag{3}
\]
Summing (1), (2), (3), and using \(x^2 + y^2 + z^2 = 3\) we have
\[
f(x, y, z) \geq \frac{4}{9} \left( \frac{1}{2} (1 + x^2) + \frac{1}{2} (1 + y^2) + \frac{1}{2} (1 + z^2) \right)
\geq \frac{8}{3} - \frac{1}{18}(3 + 2(x^2 + y^2 + z^2) + (x^2y^2 + y^2z^2 + z^2x^2)).
\]
Since clearly \(x^2y^2 + y^2z^2 + z^2x^2 \leq x^4 + y^4 + z^4\) we have
\[
3(x^2y^2 + y^2z^2 + z^2x^2) \leq (x^2 + y^2 + z^2)^2.
\]
From (4) and (5) we can conclude that
\[
f(x, y, z) \geq \frac{8}{3} - \frac{1}{18}(3 + 6 + 3) = 2.
\]
It is clear that equality holds if and only if \(x = y = z = 1\).

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SËFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy (two proofs); TITU ZVONARU, Comănești, Romania; and the proposer.


Find all quadruples \(a, b, c, d\) of positive real numbers that are solutions to
the system of equations
\[
a + b + c + d = 4,
\]
\[
\left( \frac{1}{a^{12}} + \frac{1}{b^{12}} + \frac{1}{c^{12}} + \frac{1}{d^{12}} \right) (1 + 3abcd) = 16.
\]

Solution by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

By the Arithmetic-Geometric Means Inequality,
\[
abcd \leq \left( \frac{a + b + c + d}{4} \right)^4 = 1,
\]
with equality if and only if \(a = b = c = d = 1\). Also,
\[
16 = \left( \frac{1}{a^{12}} + \frac{1}{b^{12}} + \frac{1}{c^{12}} + \frac{1}{d^{12}} \right) (1 + 3abcd)
\geq \left( \frac{4}{a^4b^4c^4d^4} \right) (4abcd) = \frac{16}{a^2b^2c^2d^2} \geq 16.
\]

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Since equality must hold throughout, \( a = b = c = d = 1 \).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIC, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; OLIVER GEUPEL, Brühl, NRW, Germany; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; TITU ZVONARU, Comănești, Romania; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer.


Let \( a, b, c \) and \( r \) be the edge-lengths and the inradius of a triangle \( ABC \). Find the minimum value of the expression

\[
E = \left( \frac{a^2 b^2}{a + b - c} + \frac{b^2 c^2}{b + c - a} + \frac{c^2 a^2}{c + a - b} \right) r^{-3}.
\]

I. Solution by Titu Zvonar, Comănești, Romania.

We shall see that the minimum value of \( E \) is \( 72\sqrt{3} \), its value when \( \Delta ABC \) is equilateral. Let \( F \) be the area of the triangle, and \( s = \frac{1}{2}(a+b+c) \) be its semiperimeter. It is known that \( F = sr \), \( s \geq 3\sqrt{3} \) (item 5.11 in [1]), and \( ab + bc + ca \geq 4F\sqrt{3} \) (item 4.5 in [1]). Applying the Cauchy-Schwarz inequality to the vectors

\[
\left( \frac{ab}{\sqrt{a+b-c}}, \frac{bc}{\sqrt{b+c-a}}, \frac{ca}{\sqrt{c+a-b}} \right) \quad \text{and} \quad \left( \sqrt{a+b-c}, \sqrt{b+c-a}, \sqrt{c+a-b} \right),
\]

we have

\[
E \geq \frac{(ab + bc + ca)^2}{a + b - c + b + c - a + c + a - b} \cdot r^{-3} = \frac{(ab + bc + ca)^2}{2s} \cdot \frac{1}{r} \cdot \frac{s^2}{F^2} = \frac{(ab + bc + ca)^2}{2} \cdot \frac{1}{F^2} \cdot \frac{s}{r} \geq 16F^2 \cdot \frac{3}{2} \cdot \frac{1}{F^2} \cdot \frac{3\sqrt{3}}{r} = 72\sqrt{3}.
\]

In an equilateral triangle we have \( a = b = c \), and \( r = \frac{a}{2\sqrt{3}} \); for these values \( E \) attains its lower bound, namely \( 72\sqrt{3} \).

II. Solution by Kee-Wai Lau, Hong Kong, China.

We use the notation of the previous solution; along with the inequality \( s \geq 3\sqrt{3}r \), we use here Euler’s inequality \( R \geq 2r \) together with \( F = \sqrt{s(s-a)(s-b)(s-c)} = rs \) and \( abc = 4srR \) (where \( R \) is the circumradius). By the AM-GM inequality we have

\[
E \geq 3 \left( \frac{a^4 b^4 c^4}{(a + b - c)(b + c - a)(c + a - b)} \right)^{1/3} r^{-3}.
\]

Using our equations for \( F \) and for \( abc \), we see that the right-hand side of this
inequality equals
\[
3 \left( \frac{a^4b^4c^4s^4}{8F^2r^3} \right)^{1/3} = 3 \left( \frac{32s^3R^4}{r^3} \right)^{1/3}.
\]
By the two inequalities referred to above we obtain
\[
3 \left( \frac{32s^3R^4}{r^3} \right)^{1/3} \geq 72\sqrt{3}.
\]
Thus \(E \geq 72\sqrt{3}\). Equality holds when triangle \(ABC\) is equilateral; therefore, the minimum value of \(E\) is \(72\sqrt{3}\).

References


Also solved by ARKADY ALT, San Jose, CA, USA; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; SEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; JOHN G. HEUVER, Grande Prairie, AB; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

Most of the submitted solutions were quite similar to a featured solution.


Find the greatest positive integer \(m\) such that \(2^m\) divides
\[
2011^{(2013^{2016}) - 1} - 1.
\]
Solution by Roy Barbara, Lebanese University, Fanar, Lebanon.

For a positive integer \(x\), let \(e(x)\) denote the exponent of 2 in the prime factorization of \(x\). Clearly, \(e(x_1x_2x_3\cdots x_r) = e(x_1) + e(x_2) + e(x_3) + \cdots + e(x_r)\).

We will use the following properties of \(e\) to solve the stated problem.

If \(a\) and \(n\) are odd, then
\[
e (a^n - 1) = e(a - 1).
\]

If \(a\) is odd and \(n\) is even, then
\[
e (a^n + 1) = 1.
\]

Proof of (1): Since \((a^n - 1) = (a - 1)(a^{n-1} + a^{n-2} + \cdots + a + 1)\) and the right factor is an odd sum of odd integers, hence it is odd and the result follows.

Proof of (2): Set \(n = 2t\). As \(a\) is odd, \(a^2 \equiv 1 \pmod{4}\), so \(a^n + 1 \equiv (a^2)^t + 1 \equiv 2 \pmod{4}\) and the result follows.
We will use the above results to show that $m = 9$ for the given problem. 

Set $\omega = 2011$, $\theta = 2013$ and $A = \theta^{2016} - 1$. The number from the problem is $N = \omega^A - 1$. 

We have $A = (\theta^{32})^{63} - 1$. By (1) we get $e(A) = e(\theta^{32} - 1)$. Factoring yields 

$$\theta^{32} - 1 = \left( \theta^{16} + 1 \right) \left( \theta^8 + 1 \right) \left( \theta^4 + 1 \right) \left( \theta^2 + 1 \right) (\theta + 1)(\theta - 1).$$

From (2) we get $e(\theta^{16} + 1) = e(\theta^8 + 1) = e(\theta^4 + 1) = e(\theta^2 + 1) = 1$. Furthermore, $e(\theta + 1) = e(2 \times 1007) = 1$ and $e(\theta - 1) = e(4 \times 503) = 2$. Hence, 

$$e(A) = e(\theta^{32} - 1) = 1 + 1 + 1 + 1 + 1 + 2 = 7.$$

Therefore, we can set $A = 2^7 B = 128 B$ for some odd number $B$. 

Now, $N = (\omega^{128})^B - 1$ so, by (1), $e(N) = e(\omega^{128})$. Factoring yields 

$$\omega^{128} - 1 = \left( \omega^{64} + 1 \right) \left( \omega^{32} + 1 \right) \left( \omega^{16} + 1 \right) \left( \omega^8 + 1 \right) \left( \omega^4 + 1 \right) \left( \omega^2 + 1 \right) (\omega+1)(\omega-1).$$

From (2) we get $e(\omega^{64} + 1) = e(\omega^{32} + 1) = e(\omega^{16} + 1) = e(\omega^8 + 1) = e(\omega^4 + 1) = e(\omega^2 + 1) = 1$. Furthermore, $e(\omega+1) = e(4 \times 503) = 2$ and $e(\omega-1) = e(2 \times 1005) = 1$.

Finally, 

$$m = e(N) = e(\omega^{128} - 1) = 1 + 1 + 1 + 1 + 1 + 2 + 1 = 9$$

and we are done. 

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JENNIFER DEMPSEY and MICHAEL MURPHY, St. Bonaventure University, St. Bonaventure, NY, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. One incorrect solution was received. 

3630. [2011 : 170, 173] Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA. 

Let $a, b$, and $c$ be nonnegative real numbers such that $a + b + c = 3$. Prove that 

$$\frac{ab(b + c)}{2 + c} + \frac{bc(c + a)}{2 + a} + \frac{ca(a + b)}{2 + b} \leq 2.$$ 

Solution by Titu Zvonaru, Comănești, Romania, modified and expanded by the editor. 

Note first that 

$$\sum_{cyclic} ab^2 c = abc(a + b + c) = 3abc. \quad (1)$$

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Clearing the denominators and using (1), the given inequality is equivalent, in succession, to:

\[
\sum_{\text{cyclic}} (ab^2 + abc)(ab + 2a + 2b + 4) \leq (ab + 2a + 2b + 4)(4 + 2c)
\]

\[
\sum_{\text{cyclic}} (a^2b^3 + a^2b^2 + 2a^2b^2 + 2a^2bc + 2ab^3 + 2ab^2c + 4ab^2 + 4abc)
\]

\[
\leq 4ab + 8a + 8b + 16 + 2abc + 4ac + 4bc + 8c
\]

\[
\sum_{\text{cyclic}} a^2b^2 + abc \sum_{\text{cyclic}} ab + 2 \sum_{\text{cyclic}} a^2b^2 + 2 \sum_{\text{cyclic}} ab^3 + \sum_{\text{cyclic}} 4ab^2 + 24abc
\]

\[
\leq 40 + 4 \sum_{\text{cyclic}} ab + 2abc
\]

\[
\sum_{\text{cyclic}} a^2b^2 + abc \sum_{\text{cyclic}} ab + 2 \sum_{\text{cyclic}} a^2b^2 + 2 \sum_{\text{cyclic}} ab^3 + \sum_{\text{cyclic}} 4ab^2 + 22abc
\]

\[
\leq 40 + 4 \sum_{\text{cyclic}} ab.
\] (2)

Since

\[
(a + b + c)^3 - 3(ab + bc + ca) = \frac{1}{2} \left( (a - b)^2 + (b - c)^2 + (c - a)^2 \right) \geq 0
\]

we have \(3(ab + bc + ca) \leq (a + b + c)^2 = 9\), so

\[
\sum_{\text{cyclic}} ab \leq 3.
\] (3)

Furthermore, in the published solution to Crux problem 3527 [2011 : 177] (proposed by the same proposer of the current problem), George Apostolopoulos proved that if \(a, b,\) and \(c\) are nonnegative real such that \(a + b + c = 3\), then

\[
ab^2 + bc^2 + ca^2 + abc \leq 4.
\] (4)

Using (1) and (4) we obtain successively that

\[
\sum_{\text{cyclic}} a^2b^2 + \sum_{\text{cyclic}} ab^3 + \sum_{\text{cyclic}} abc(a + b + c) \leq 12
\]

\[
\sum_{\text{cyclic}} a^2b^2 + \sum_{\text{cyclic}} ab^3 + 6abc \leq 12.
\] (5)

From (4) and (5) we see that in order to establish (2), it suffices to show that

\[
\sum_{\text{cyclic}} a^2b^2 + abc \sum_{\text{cyclic}} ab + 6abc \leq 4 \sum_{\text{cyclic}} ab.
\] (6)

[Ed.: Note that (2) would follow if we added 4(4) + 2(5) to (6).]
Now (6) is equivalent, in succession, to

$$\sum_{\text{cyclic}} (a^2b^3 + ab^2c^2 + a^3bc + a^2b^2c) + 6abc \leq 4 \sum_{\text{cyclic}} ab + \sum_{\text{cyclic}} (ab^2c^2 + a^3bc)$$

$$\sum_{\text{cyclic}} ab(ab^2 + bc^2 + ca^2 + abc) + 6abc \leq 4 \sum_{\text{cyclic}} ab + abc \sum_{\text{cyclic}} (a^2 + ab) \quad (7)$$

Since $$ab^2 + bc^2 + ca^2 + abc \leq 4$$ by (4), it remains to show that $$6abc \leq abc \sum_{\text{cyclic}} (a^2 + ab)$$ or $$\sum_{\text{cyclic}} (a^2 + ab) \geq 6$$ which is true since

$$\sum_{\text{cyclic}} (a^2 + ab) = \sum_{\text{cyclic}} a^2 + \sum_{\text{cyclic}} ab = (a + b + c)^2 - \sum_{\text{cyclic}} ab = 9 - \sum_{\text{cyclic}} ab \geq 6$$

by (3). This establishes (7) and completes the proof.

If equality holds, then we have $$a = b = c$$ or $$abc = 0$$ and $$ab^2 + bc^2 + ca^2 = 4$$.

Suppose $$c = 0$$, then solving $$a + b = 3$$ and $$ab^2 = 4$$ we obtain $$a(3 - a)^2 = 4$$ or $$a^3 - 6a^2 + 9a - 4 = 0$$ or $$(a + 2)^2(a - 1) = 0$$ so $$a = 1$$ and $$b = 2$$. Hence, either $$a = b = c = 1$$ or $$(a, b, c) = (1, 2, 0), (0, 1, 2),$$ or $$(2, 0, 1)$$. It is readily checked that all these ordered triples yield equality.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. There were also two submitted solutions which either had errors or contained claims without proof.