OC71. On définit une suite \( a_n \) d’entiers positifs avec \( a_1 = 1 \) et \( a_{n+1} \) comme étant le plus petit entier tel que
\[ \text{ppcm}(a_1, ..., a_{n+1}) > \text{ppcm}(a_1, ..., a_n). \]
Trouver l’ensemble \( \{a_n | n \in \mathbb{Z}\} \).

OC72. Montrer qu’il y a une infinité d’entiers positifs tels que les moyennes arithmétique et géométrique de leurs diviseurs sont des entiers.

OC73. Trouver toutes les suites non décroissantes \( a_1, a_2, a_3, ... \) de nombres naturels telles pour chaque \( i, j \in \mathbb{N}, i + j \) et \( a_i + a_j \) ont le même nombre de diviseurs.

OC74. Soit \( H \) l’orthocentre d’un triangle acutangle \( ABC \) avec \( \Gamma \) comme cercle circonscrit. Soit \( P \) un point sur l’arc \( BC \) (ne contenant pas \( A \)) de \( \Gamma \), et soit \( M \) un point sur l’arc \( CA \) (ne contenant pas \( B \)) de \( \Gamma \) de sorte que \( H \) soit sur le segment \( PM \). Soit \( K \) un autre point sur \( \Gamma \) de telle sorte que \( KM \) soit parallèle à la droite de Simson de \( P \) par rapport au triangle \( ABC \). Soit \( J \) le point d’intersection des segments \( BC \) et \( KQ \). Montrer que \( \triangle KJM \) est un triangle isocèle.

OC75. Soit \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \) et \( Q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_0 \) deux polynômes à coefficients entiers tels que \( a_n - b_n \) est un nombre premier, \( a_nb_0 - a_0b_n \neq 0 \), et \( a_{n-1} = b_{n-1} \). Supposons qu’il existe un nombre rationnel \( r \) tel que \( P(r) = Q(r) = 0 \). Montrer que \( r \in \mathbb{Z} \).

OLYMPIAD SOLUTIONS

OC11. For non-empty subsets \( A, B \subseteq \mathbb{Z} \) define \( A + B \) and \( A - B \) by
\[ A + B = \{a + b \mid a \in A, b \in B\}, \ A - B = \{a - b \mid a \in A, b \in B\}. \]
In the sequel we work with non-empty finite subsets of \( \mathbb{Z} \).
Prove that we can cover \( B \) by at most \( \frac{|A + B|}{|A|} \) translates of \( A - A \), i.e. there exists \( X \subseteq \mathbb{Z} \) with \(|X| \leq \frac{|A + B|}{|A|} \) such that
\[ B \subseteq \bigcup_{x \in X} (x + (A - A)) = X + A - A. \]
(Originally question #1 from the 1st selection test, 60th National Mathematical Olympiad Selection Tests for the Balkan and IMO, by Imre Rusza, Hungary.)

Solved by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.
For a subset \( X \subseteq B \), consider the family \( F_X := \{A + x | x \in X\} \). Since \( B \) is finite, we can find a subset \( X \subseteq B \) so that the elements of \( F_X \) are pairwise disjoint, and \( X \) is maximal with this property, that is:
(1) For all $x, y \in X$ with $x \neq y$ we have $(x + A) \cap (y + A) = \emptyset$.

(2) For each $b \in B$ there exists some $x \in X$ so that $(x + A) \cap (b + A) \neq \emptyset$.

We claim that this $X$ has the required properties.

First, by (1) we have

$$|X| \cdot |A| = \sum_{x \in X} |x + A| = |A + X| \leq |A + B|.$$ 

Thus

$$|X| \leq \frac{|A + B|}{|A|}.$$ 

Now, let $b \in B$. Then, by the second condition there exists some $x \in A$ such that

$$(b + A) \cap (x + A) \neq \emptyset.$$ 

Let $z \in (b + A) \cap (x + A)$. Then, there exists $a, a' \in A$ so that

$$z = b + a = x + a'.$$

Thus

$$b = x + a' - a \in X + (A - A),$$

which completes the proof.

**OC12.** Let $k$ be a positive integer greater than 1. Prove that for every non-negative integer $m$ there exist $k$ positive integers $n_1, n_2, \ldots, n_k$, such that

$$n_1^2 + n_2^2 + \cdots + n_k^2 = 5^{m+k}.$$ 

(Originally question #2 from the 53rd National Mathematical Olympiad in Slovenia, 2nd Selection Exam for the IMO 2009.)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany and Titu Zvonaru, Comănești, Romania. We give the solution of Geupel.

We prove the statement by induction on $k$.

**P(2):** If $m = 2q$ then we have

$$(3 \cdot 5^q)^2 + (4 \cdot 5^q)^2 = 5^{m+2}.$$ 

If $m = 2q - 1$ then we have

$$(5^q)^2 + (2 \cdot 5^q)^2 = 5^{m+2}.$$ 

**P(3):** If $m = 2q$ then we have

$$(3 \cdot 5^q)^2 + (4 \cdot 5^q)^2 + (10 \cdot 5^q)^2 = 5^{m+3}.$$ 

If $m = 2q - 1$ then we have

$$(12 \cdot 5^{q-1})^2 + (15 \cdot 5^{q-1})^2 + (16 \cdot 5^{q-1})^2 = 5^{m+3}.$$ 

Now we show that $P(k) \Rightarrow P(k + 2)$:

By $P(2)$, we can find $n_1, n_2$ so that

$$n_1^2 + n_2^2 = 5^{m+k+1}.$$ 

Also, by $P(k)$ we can find $n_3, \ldots, n_{k+2}$ so that

$$n_3^2 + \ldots + n_{k+2}^2 = 5^{m+k+1}.$$ 

Then

$$n_1^2 + n_2^2 + (2n_3)^2 + \ldots + (2n_{k+2})^2 = 5^{m+k+1} + 4 \cdot 5^{m+k+1} = 5^{m+k+2}.$$ 

**OC13.** Let $ABC$ be an acute triangle and let $D$ be a point on the side $AB$. The circumcircle of the triangle $BCD$ intersects the side $AC$ at $E$. The circumcircle of the triangle $ADC$ intersects the side $BC$ at $F$. Let $O$ be the circumcentre of triangle $CEF$. Prove that the points $D$ and $O$ and the circumcentres of the triangles $ADE$, $ADC$, $DBF$ and $DBC$ are concyclic and the line $OD$ is perpendicular to $AB$. (Originally question #2 from the 53rd National Mathematical Olympiad in Slovenia, 1st Selection Exam for the IMO 2009.)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany and Titu Ţzonaru, Comăneşti, Romania. We give the solution of Geupel.

Let us denote by $\alpha = \angle BAC, \beta = \angle CBA, \gamma = \angle ACB$. Let $O_{ADE}, O_{BDF}, O_{ACFD},$ and $O_{BCED}$ denote the circumcircles of $\triangle ADE$, $\triangle BDF$, quadrilateral $ACFD$, and quadrilateral $BCED$, respectively. Let $O_{CEF} := O$. The points $O_{ACFD}$ and $O_{CEF}$ are on the perpendicular bisector of $CF$, thus $O_{ACFD}O_{CEF} \perp CF$. Similarly, $O_{BCED}O_{CEF} \perp CE$. 

Copyright © Canadian Mathematical Society, 2013
Since $\angle ECF = \gamma$, we have

$$\angle O_{ACFD}O_{CEF}O_{BCED} \in \{\gamma, 180^\circ - \gamma\}. \quad (1)$$

We have $O_{ACFD}O_{ADE} \perp AD$ and $O_{BCED}O_{ADE} \perp DE$. Since $B, C, D,$ and $E$ are concyclic, we have $\angle ADE = 180^\circ - \angle BDE = BCE = \gamma$. Thus,

$$\angle O_{ACFD}O_{ADE}O_{BCED} \in \{\gamma, 180^\circ - \gamma\}. \quad (2)$$

Similarly,

$$\angle O_{ACFD}O_{BDF}O_{BCED} \in \{\gamma, 180^\circ - \gamma\}. \quad (3)$$

Combining (1), (2), and (3), we get that the points $O_{ADE}, O_{BDF}, O_{CEF}, O_{ACFD},$ and $O_{BCED}$ are on a common circle $\Gamma$. We now prove that $D \in \Gamma$.

Since $O_{ACFD}O_{ADE} \perp AD$ and $O_{BCED}O_{BDF} \perp BD$, we have $O_{ACFD}O_{ADE} \parallel O_{BCED}O_{BDF}$.

Thus, the chords $O_{ACFD}O_{BCEAD}$ and $O_{ADE}O_{BDF}$ of the circle $\Gamma$ are congruent.

Moreover, in $\triangle ADE$, we have $\angle ADO_{ADE} = 90^\circ - \angle AED = 90^\circ - \beta$.

Similarly, $\angle BDO_{BDF} = 90^\circ - \alpha$. Thus,

$$\angle ADO_{ADE}O_{BDF} = 180^\circ - \angle ADO_{ADE} - \angle BDO_{BDF} = \gamma,$$

so that $D \in \Gamma$.

We now show that $\angle ADO_{CEF} = 90^\circ$.

In $\triangle CEF$, we have $\angle EOC_{CEF}F = 2\gamma$. Since

$$\angle EDF = 180^\circ - \angle ADE - \angle BDF = 180^\circ - 2\gamma,$$

the points $D, E, F, O_{CEF}$ are concyclic.

By $EO_{CEF} = FO_{CEF}$, we deduce the corresponding arcs are equal, thus

$$\angle EDO_{CEF} = \angle EDF/2 = 90^\circ - \gamma.$$

Consequently,

$$\angle ADO_{CEF} = \angle ADE + \angle EDO_{CEF} = \gamma + (90^\circ - \gamma) = 90^\circ.$$

**OC14.** Let $a_n, b_n, n = 1, 2, \ldots$ be two sequences of integers defined by $a_1 = 1, b_1 = 0$ and for $n \geq 1$,

$$a_{n+1} = 7a_n + 12b_n + 6,$$

$$b_{n+1} = 4a_n + 7b_n + 3.$$

Prove that $a_n^2$ is the difference of two consecutive cubes. (Originally question #2 from the Singapore Mathematical Olympiad 2010, Open Section, Round 2.)

*Solved by Arkady Alt, San Jose, CA, USA; Michel Bataille, Rouen, France; Felix Boos, University of Kaiserslautern, Kaiserslautern, Germany; Oliver*

_Crux Mathematicorum_, Vol. 38(3), March 2012
We give the solution of Song and Wang.

We prove by induction that
\[ a_n^2 = (b_n + 1)^3 - b_n^3 = 3b_n^2 + 3b_n + 1. \]

The initial step is obvious. We now proceed to proving the inductive step.

\[ a_{n+1} - (3b_{n+1}^2 + 3b_{n+1} + 1) \]
\[ = (7a_n + 12b_n + 6)^2 - 3(4a_n + 7b_n + 3)^2 - 3(4a_n + 7b_n + 3) - 1 \]
\[ = (49a_n^2 + 144b_n^2 + 168a_nb_n + 84a_n + 144b_n + 36) \]
\[ - 3(16a_n^2 + 49b_n^2 + 56a_nb_n + 24a_n + 42b_n + 9) - 3(4a_n + 7b_n + 3) - 1 \]
\[ = a_n^2 - 3b_n^2 - 3b_n - 1 = 0 \]

which completes the proof.

**OC15.** A ruler of length \( \ell \) has \( k \geq 2 \) marks at positions \( a_i \) units from one of the ends with \( 0 < a_1 < \cdots < a_k < \ell \). The ruler is called a Golomb ruler if the lengths that can be measured using the marks on the ruler are consecutive integers starting with 1, and each such length be measurable between a unique pair of marks on the ruler. Find all Golomb rulers. (Originally question \#4 from the 60th National Mathematical Olympiad Selection Tests for the Balkan and IMO, 2nd selection test, by Barbu Berceanu.)

Solved by Oliver Geupel, Brühl, NRW, Germany.

For simplicity let us identify a Golomb ruler \( G \) with the set \( \{a_1, \ldots, a_k\} \) of positions of its marks. Since \( G \) can measure the lengths 1, \ldots, \( \binom{k}{2} \) by definition, we have \( a_k = a_1 + \binom{k}{2} \).

Further, we have either \( a_2 = a_1 + 1 \) or \( a_k = a_{k-1} + 1 \), since the length \( \binom{k}{2} - 1 \) can be measured.

Let us call the Golomb ruler \( G' = \{a'_1, \ldots, a'_k\} \) a shift of \( G \) if \( a'_i = a_i + d \) for \( 1 \leq i \leq k \) and a constant \( d \).

Also let us call \( G' \) the reflection of \( G \) if \( a'_i = a_k + 1 - a_{k+1-i} \) for \( 1 \leq i \leq k \).

We will say that \( G \) and \( G' \) are equivalent if \( G' \) is a shift of \( G \) or of the reflection of \( G \). Equivalence, in fact, is an equivalence relation on the set of all Golomb rulers.

For each equivalence class there exists a unique member such that \( a_1 = 1 \) and \( a_2 = 2 \). We will call this Golomb ruler canonical, and we will determine it.

We first claim that there exists no Golomb ruler with \( k > 4 \).

Let's assume by contradiction that there exists a Golomb ruler with \( k > 4 \). Then there exists a canonical Golomb ruler \( G \) equivalent to it. Let's look at \( G \).

For simplicity, let's denote \( n := \binom{k}{2} \). We know that \( n \geq 10 \) and \( 1, 2, n+1 \in G \).
We know that \( n - 2 \) can be measured with \( G \). The only ways we can get \( n - 2 \) is \((n - 1) - 1\) or \( n - 2 \) or \((n + 1) - 3\). It is not possible to have \( 3 \in G \) or \( n \in G \), because in this case we would get two ways of measuring 1. Thus, we must obtain measure \( n - 2 \) between 1 and \( n - 1 \), and hence
\[
\{1, 2, n - 1, n + 1\} \subset G.
\]
Now, \( n - 4 \) can also be measured with \( G \). Thus, one of the pairs \((1, n - 3)\), \((2, n - 2)\), \((3, n - 1)\), \((4, n)\) or \((5, n + 1)\) must appear in \( G \). Again, by unique measurability, we cannot have \( 3, 4, n - 2, n - 3 \) in \( G \) [Note that \( 4 < n - 2 \)]. Thus \( 5 \in G \).

So far we have
\[
\{1, 2, 5, n - 1, n + 1\} \subset G.
\]

**Case 1:** \( k = 5 \). Then \( n = 10 \) and our \( G \) is \( G = \{1, 2, 5, 9, 11\} \). But this is not a Golomb ruler, since \( 9 - 5 = 5 - 1 \).

**Case 2:** \( k \geq 6 \). Since the length \( n - 5 \) can be measured using \( G \), at least one of the pairs \( \{1, n - 4\} \), \( \{2, n - 3\} \), \( \{3, n - 2\} \), \( \{4, n - 1\} \), \( \{5, n\} \), and \( \{6, n + 1\} \) must be contained in \( G \). Moreover \( 6 < n - 4 \).

But for reasons of unique measurability, the numbers \( n - 4, n - 3, n - 2, 4, n \), and 6 are not elements of \( G \). This is a contradiction, as desired.

We know now that \( k \leq 4 \).

If \( k = 2 \), the only canonical Golomb ruler is \( \{1, 2\} \).

If \( k = 3 \), then \( \binom{3}{2} + 1 = 4 \), thus the only canonical Golomb ruler is \( \{1, 2, 4\} \).

If \( k = 4 \), then \( \binom{4}{2} + 1 = 7 \), thus any canonical Golomb ruler must contain \( \{1, 2, 7\} \) and one more number. Let that number be \( x \). The only ways to measure a segment of length 4 is if \( x - 1 = 4 \) or \( x - 2 = 4 \) or \( 7 - x = 4 \), and it is easy to see that only \( x = 5 \) creates a Golomb ruler.

Thus, the only canonical Golomb Rulers are
\[
\{1, 2\} ; \{1, 2, 4\} ; \{1, 2, 5, 7\}.
\]

This means that all the Golomb Rulers are the following
\[
\{d, d + 1\} ; \{d, d + 1, d + 3\} ; \{d, d + 2, d + 3\} ; \{d, d + 1, d + 4, d + 6\} ; \{d, d + 2, d + 5, d + 6\} | d > 0.
\]