MAYHEM SOLUTIONS

Mathematical Mayhem *is being reformatted as a stand-alone mathematics journal for high school students. Solutions to problems that appeared in the last volume of* **Crux** *will appear in this volume, after which time* **Mathematical Mayhem** *will be discontinued in* **Crux**. *New* **Mayhem** *problems will appear when the journal is relaunched in 2013.*

M482. *Proposed by the Mayhem Staff.*

Using four sticks with lengths of 1 cm, 2 cm, 3 cm, and 5 cm, respectively, you can measure any integral length from 1 cm to 10 cm. Note that a stick may only be used once in a particular measurement, so the 1 cm, 2 cm, and 3 cm sticks could be used to measure 6 cm, but not the 3 cm stick twice.

(a) Find a set of ten stick lengths that can be used to represent any integral length from 1 cm to 100 cm.

(b) What is the fewest number of sticks that are needed to represent any integral length from 1 cm to 100 cm?

Solution by Florencio Cano Vargas, Inca, Spain.

(a) We can apply a similar reasoning to the one given in the problem to measure the tens: with sticks of 10, 20, 30 and 50 cm we can measure those lengths from 10 to 100 which are multiples of 10. Therefore, if we combine these sticks with the set of 1, 2, 3 and 5 cm (to measure any length from 1 to 10) it is clear that we cover all the integral distances between 1 and 100 cm. To meet the condition stated in the problem we can add two more sticks of, say 40 and 60 cm. Then the required set of sticks is (in cm units):

$$\{1, 2, 3, 5, 10, 20, 30, 40, 50, 60\}$$

(b) We will show that the minimal number of sticks is seven and we generalize the method to any length. Let l_i represent the length (in cm) of the i^{th} stick and we order them such that $l_i < l_j$ if $i < j$. Clearly we start with $l_1 = 1$ and $l_2 = 2$; with which, we can measure any length up to 3. Then we choose $l_3 = l_1 + l_2 + 1 = 4$. With l_1, l_2 and l_3 we can measure up to length 7, so that we choose $l_4 = 8$. In general, we choose

$$l_{n+1} = \sum_{k=1}^{n} l_k + 1 \qquad l_1 = 1$$

and with the set $\{l_1, ..., l_n\}$ will be able to measure lengths up to $L = \sum_{k=1}^{n} l_k$. It is easy to show by induction that an expression for the length of the sticks is

$l_i = 2^{i-1}$ from which we can calculate the lengths we can measure with n sticks, that is:

$$\sum_{k=1}^{n} l_k = 2^n - 1.$$

By imposing that $2^n - 1 > 100$ we have that the minimal value for n is 7. Therefore, the required set is:

$$\{1, 2, 4, 8, 16, 32, 64\}$$

As a matter of fact, we can measure lengths up to $2^7 - 1 = 127$ cm.

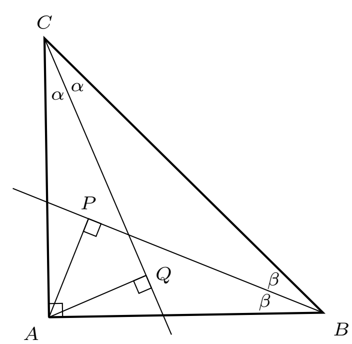
Also solved by GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; THARIQ SURYA GUMELAR, student, SMPN 8, Yogyakarta, Indonesia; JUZ'AN NARI HAIFA, student, SMPN 8, Yogyakarta, Indonesia; HASNA NADILA, student, SMPN 8, Yogyakarta, Indonesia; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; LUÍS SOUSA, ISQAPAVE, Angola; and DEXTER WEI and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON. Five incorrect solutions were submitted.

M483. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Triangle ABC has $\angle BAC = 90°$. The feet of the perpendiculars from A to the internal bisectors of $\angle ABC$ and $\angle ACB$ are P and Q, respectively. Determine the measure of $\angle PAQ$.

Solution by Gusnadi Wiyoga, student, SMPN 8, Yogyakarta, Indonesia.

Let $\angle ABC = 2\beta$ and $\angle ACB = 2\alpha$. It is clear that $2\beta + 2\alpha = 90°$, and hence $\beta + \alpha = 45°$. Now consider ΔABP. Since $\angle ABP = \frac{1}{2}\angle ABC = \beta$, then $\angle PAB = 90° - \beta$. Similarly, by considering ΔACQ we get $\angle CAQ = 90° - \alpha$. We now obtain:



$$\begin{aligned}\angle PAQ &= \angle PAB - \angle QAB \\ &= \angle PAB - [\angle CAB - \angle CAQ] \\ &= (90° - \beta) - [90° - (90° - \alpha)] \\ &= 90° - (\beta + \alpha) \\ &= 90° - 45° \\ &= 45°\end{aligned}$$

The measure of $\angle PAQ$ is therefore $45°$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; VIJAYA PRASAD NULLURI, Rajahmundry, India; RICARD

PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON.

M484. *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Solve the equation

$$x^2 + 4\left(\frac{x}{x-2}\right)^2 = 45\,.$$

Solution by Cássio dos Santos Sousa, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil.

Rewriting the equation, we obtain:

$$x^2 + \left(\frac{2x}{x-2}\right)^2 = 45.$$

Completing the square on the left hand side yields:

$$\begin{aligned} x^2 + \left(\frac{2x}{x-2}\right)^2 + \frac{4x^2}{x-2} &= 45 + \frac{4x^2}{x-2} \\ \left(x + \frac{2x}{x-2}\right)^2 &= 45 + \frac{4x^2}{x-2} \\ \left(\frac{x^2}{x-2}\right)^2 &= 45 + 4\left(\frac{x^2}{x-2}\right). \end{aligned}$$

We can take this expression, rearrange it and complete the square again to get:

$$\begin{aligned} \left(\frac{x^2}{x-2}\right)^2 - 4\left(\frac{x^2}{x-2}\right) + 4 &= 45 + 4 \\ \left(\frac{x^2}{x-2} - 2\right)^2 &= 49, \end{aligned}$$

hence, as $x \neq 2$,

$$\frac{x^2}{x-2} - 2 = 7 \Rightarrow x^2 - 9x + 18 = 0, \tag{1}$$

$$\frac{x^2}{x-2} - 2 = -7 \Rightarrow x^2 + 5x - 10 = 0. \tag{2}$$

Finally, from (1) we get the first two roots $x_1 = 6$, $x_2 = 3$ and from (2) we get the last two roots $x_3 = \dfrac{-5+\sqrt{65}}{2}$ and $x_4 = \dfrac{-5-\sqrt{65}}{2}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; SCOTT BROWN, Auburn University, Montgomery, AL, USA; FLORENCIO CANO VARGAS, Inca, Spain; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; MUHAMMAD HAFIZ FARIZI, student, SMPN 8, Yogyakarta, Indonesia; IAN JUNE L. GARCES, Ateneo *de Manila University, Quezon City, The Philippines; THARIQ SURYA GUMELAR, student, SMPN 8, Yogyakarta, Indonesia; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; MITEA MARIANA, No. 2 Secondary School, Cugir, Romania; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON (2 solutions); GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One incomplete solution was received.*

With the exception of Mariana and the proposer, all other solvers expanded and rearranged the equation to get $x^4 - 4x^3 - 37x^2 + 180x - 180 = 0$, *which factors to* $(x-3)(x-6)(x^2+5x-10) = 0$, *from which the roots are easily extracted.*

M485. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Prove that

$$\prod_{k=1}^{n} \binom{n}{k} = \frac{1}{n!} \prod_{k=1}^{n} \frac{k^k}{(n-k)!}$$

for all $n \in \mathbb{N}$.

Solution by Geneviève Lalonde, Massey, ON.

By direct computation we get

$$\prod_{k=1}^{n} \binom{n}{k} = \prod_{k=1}^{n} \frac{n!}{k!(n-k)!}$$

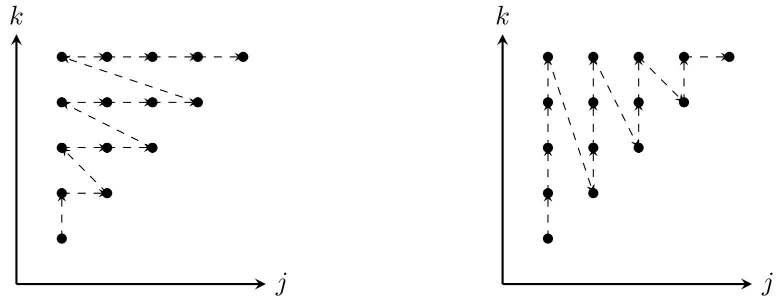
but since these are all products, we can rewrite this as

$$\prod_{k=1}^{n} \binom{n}{k} = \frac{\displaystyle\prod_{k=1}^{n} n!}{\left(\displaystyle\prod_{k=1}^{n} k!\right)\left(\displaystyle\prod_{k=1}^{n} (n-k)!\right)}. \tag{1}$$

Working with the product in the numerator, can rewrite it as

$$\prod_{k=1}^{n} n! = \prod_{k=1}^{n} \prod_{j=1}^{n} j.$$

We can think of this product as if we are considering a number of lattice points and taking the product of their ordinates. From the given product, we travel through the points as indicated in the first diagram below (with the example $n = 5$). Since we are multiplying all these numbers together and multiplication is commutative, it doesn't matter what order we do the multiplication in. We will choose the order in the second diagram below because it will simplify to a more useful form.



Now the product can be simplified to

$$\prod_{k=1}^{n} n! = \prod_{k=1}^{n}\prod_{j=1}^{n} j = \prod_{j=1}^{n}\prod_{k=1}^{n} j = \prod_{j=1}^{n} j^n.$$

Similarly, working with one of the bottom products we get

$$\prod_{k=1}^{n} k! = \prod_{k=1}^{n}\prod_{j=1}^{k} j = \prod_{j=1}^{n}\prod_{k=j}^{n} j = \prod_{j=1}^{n} j^{n+1-j}.$$

Hence

$$\frac{\prod_{k=1}^{n} n!}{\prod_{k=1}^{n} k!} = \frac{\prod_{j=1}^{n} j^n}{\prod_{j=1}^{n} j^{n+1-j}} = \prod_{j=1}^{n} \frac{j^n}{j^{n+1-j}} = \prod_{j=1}^{n} \frac{j^j}{j} = \frac{\prod_{j=1}^{n} j^j}{\prod_{j=1}^{n} j} = \frac{\prod_{j=1}^{n} j^j}{n!},$$

which, when substituted into (1) yields the result.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; FLORENCIO CANO VARGAS, Inca, Spain; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; GUSNADI WIYOGA, student, SMPN 8, Yogyakarta, Indonesia; and the proposer. One incorrect solution was received.

M486. *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

How many distinct numbers are in the list

$$\frac{1^2-1+4}{1^2+1},\ \frac{2^2-2+4}{2^2+1},\ \frac{3^2-3+4}{3^2+1},\ \ldots,\ \frac{2011^2-2011+4}{2011^2+1}\ ?$$

Solution by Florencio Cano Vargas, Inca, Spain.

We can rewrite the general term of this sequence as

$$a_n = \frac{n^2-n+4}{n^2+1} = \frac{n^2+1-n+3}{n^2+1} = 1 - \frac{n-3}{n^2+1}.$$

If two terms, a_n and a_m, are equal then

$$1 - \frac{n-3}{n^2+1} = 1 - \frac{m-3}{m^2+1}$$

which yields

$$\begin{aligned} \frac{m-3}{m^2+1} &= \frac{n-3}{n^2+1} \\ mn^2 - 3n^2 + m - 3 &= m^2n - 3m^2 + n - 3 \\ 3m^2 - 3n^2 + m - n &= m^2n - mn^2 \\ (3m+3n+1)(m-n) &= mn(m-n). \end{aligned}$$

Since $m = n$ leads to the trivial case $a_n = a_n$, we will assume that $m \neq n$, hence

$$\begin{aligned} mn &= 3m + 3n + 1 \\ m(n-3) &= 3n + 1. \end{aligned}$$

Clearly $n \neq 3$, since m and n are integers and $n = 3 \Rightarrow 3n+1 = 0$, a contradiction. Thus

$$m = \frac{3n+1}{n-3} = \frac{3n-9+10}{n-3} = 3 + \frac{10}{n-3}.$$

Since the denominator, $n-3$, has to divide 10 we have only four possibilities:

$$\begin{aligned} n-3 = 1 &\Rightarrow n = 4, m = 13; \\ n-3 = 2 &\Rightarrow n = 5, m = 8; \\ n-3 = 5 &\Rightarrow n = 8, m = 5; \\ n-3 = 10 &\Rightarrow n = 13, m = 4. \end{aligned}$$

So there are only two pairs of terms that are equal: $a_4 = \dfrac{16}{17} = a_{13}$ and $a_5 = \dfrac{12}{13} = a_8$. Therefore there are 2009 distinct numbers in the list.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; ANTONIO LEDESMA LÓPEZ, Instituto de Educación Secundaria No. 1, Requena-Valencia, Spain; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; CÁSSIO DOS SANTOS SOUSA, Instituto Tecnológico de Aeronáutica, São Paulo, Brazil; EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, ON; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. Three incorrect solutions were received.

Note, the equation $mn = 3m+3n+1$ could be rewritten as $mn-3m-3n+9 = 10$ which could be factored as $(m-3)(n-3) = 10$. Then, using the fact that both $m-3$ and $n-3$ must be factors of 10, we come to the same conclusion as the featured solution. Also note that if we looked at the infinite *sequence $\left\{\frac{n^2-n+4}{n^2+1}\right\}_{n=1}^{\infty}$, the featured solutions shows that there are only two repeated values in the whole sequence.*

M487. *Proposed by Samuel Gómez Moreno, Universidad de Jaén, Jaén, Spain.*

Let m be a positive integer. Find all real solutions to the equation

$$m+\sqrt{m+\sqrt{m+\cdots\sqrt{m+\sqrt{m+\sqrt{x}}}}}=x,$$

in which the integer m occurs n times.

Solution by Florencio Cano Vargas, Inca, Spain.

Let us consider first the equation $m+\sqrt{x}=x$. To solve it, let us denote $t=\sqrt{x}>0$ and then $x=t^2$ and the equation to solve is $t^2-t-m=0$ which has solutions $t=\dfrac{1\pm\sqrt{1+4m}}{2}$. Since $1-\sqrt{1+4m}<1-\sqrt{1}=0$, we will keep only $t=\dfrac{1+\sqrt{1+4m}}{2}$ as t must be positive.

In order to solve the equation given in the problem we define the function $f:\mathbb{R}^+\to\mathbb{R}^+$, $f(x)=m+\sqrt{x}$. Thus we see that

$$\begin{aligned}
m+\sqrt{x}=x &\iff x=f(x),\\
m+\sqrt{m+\sqrt{x}}=x &\iff x=(f\circ f)(x)=f^2(x),\\
m+\sqrt{m+\sqrt{m+\sqrt{x}}}=x &\iff x=(f\circ f\circ f)(x)=f^3(x),\\
&\vdots\\
m+\sqrt{m+\sqrt{m+\cdots\sqrt{m+\sqrt{m+\sqrt{x}}}}}=x &\iff x=(f\circ f\circ f\circ\cdots f\circ f)(x)=f^n(x),
\end{aligned}$$

where, in the last line, the integer m appears n times.

We can easily see that if $x=f(x)$ then $f^2(x)=(f\circ f)(x)=f(x)=x$ and hence $f^n(x)=x$ for all $n\ge 1$. Thus, a solution to the equation $x=f(x)$ is also a solution to all the equations of the form $f^n(x)=x$. If we let x_0 be the solution to $x=f(x), x>0$, then

$$\sqrt{x_0}=\frac{1+\sqrt{1+4m}}{2}\Rightarrow x_0=\frac{1+2m+\sqrt{1+4m}}{2}.$$

Note that $f(x)-x$ is negative on $(0,x_0)$ and positive on (x_0,∞). Thus we can prove, by induction, that:

- if $x\in(0,t)$ then $f^n(x)<x$ and
- If $x\in(t,\infty)$ then $f^n(x)>x$.

Hence x_0 is the *only* solution to the equation $f^n(x)=x$ for all $n\ge 1$.

Also solved by by the proposer. One incorrect solution was received.