SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

478 $\star$. [1979 : 229; 1980 : 219-220; 1985 : 189-190; 1987: 151-152] *Proposed by Murray S. Klamkin, University of Alberta.*

Consider the following theorem: *If the circumcircles of the four faces of a tetrahedron are congruent, then the circumcentre O of the tetrahedron and its incentre I coincide.* An editor's comment following *Crux* 330 [1978: 264] claims that the proof of this theorem is "easy." Prove it.

V. Solution by Tomasz Cieśla, student, University of Warsaw, Poland with help from Dominik Burek, student, Jagiellonian University, Cracow, Poland.

We begin with a lemma. (See [1, Article 263], where an alternative proof can be found.)

Lemma *The points F where the face ABC of tetrahedron $ABCD$ touches the insphere, and G where it touches the exsphere opposite D, are isogonal conjugates in triangle ABC.*

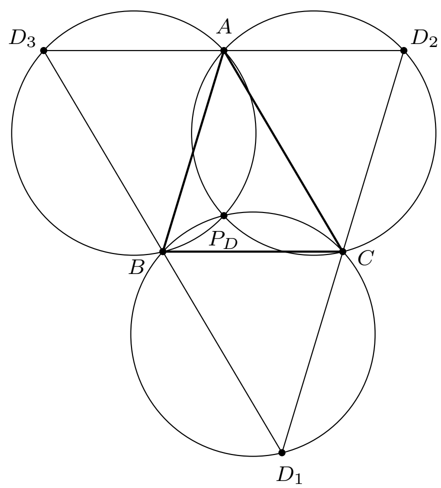
Proof of the Lemma. Consider the infinite cone with vertex D that is tangent to the faces DAB, DBC, DCA. Let e be the intersection of the face ABC with this cone; thus e is an ellipse inscribed in triangle ABC. There is a nice pictorial proof by Dandelin (in 1822) that F and G are the foci of e. (See, for example, [2, Section 17.3, page 227] or Wikipedia's article on "Dandelin Spheres." Indeed, consider any point P on e. Let DP be tangent to the insphere at X and to the exsphere at Y. Since segments PF and PX are tangent to the insphere, we have $PF = PX$. Likewise, $PG = PY$, whence $FP + PG = PX + PY = XY$, which does not depend on the choice of P on e.) Because F and G are foci of an ellipse inscribed in ΔABC, they are isogonal conjugates with respect to that triangle. (See, for example, [3, Paragraph 2323, pages 1109-1110].)

For the proof of the theorem, observe that the four finite cones having the circumcircles as base and O as vertex are congruent, whence O must be equidistant from each face of the tetrahedron; here are the details: Denote the orthogonal projections of O into faces BCD, CDA, DAB, ABC of the tetrahedron $ABCD$ by O_A, O_B, O_C, O_D, respectively. Let $R = OA = OB = OC = OD$. By Pythagoras we have $R^2 - OO_A^2 = O_AB^2 = O_AC^2 = O_AD^2$, thus O_A is the circumcentre of BCD. Analogously, O_B, O_C, O_D are the circumcentres of CDA, DAB, ABC. Let the common radius of the congruent circumcircles of these faces equal r. Then $R^2 - r^2 = OO_A^2 = OO_B^2 = OO_C^2 = OO_D^2$, whence O is equidistant from faces of tetrahedron $ABCD$; that is, O coincides with either the incentre or an excentre of tetrahedron $ABCD$. A tetrahedron can have two types of excentres. We must show that it is not possible for the circumcentre of $ABCD$ to coincide with either type of excentre.

Assume first that O coincides with the excentre opposite vertex D. Then O_D is the point where that exsphere is tangent to face ABC. Denote the points where the insphere touches faces BCD, CDA, DAB, and ABC by P_A, P_B, P_C, and P_D, respectively. By the lemma, P_D and O_D are isogonal conjugates in triangle ABC; consequently, because O_D is its circumcentre, so P_D is its orthocentre.

In the circle BCD we have $\sin\angle BDC = \frac{BC}{2r} = \sin\angle BAC$; thus either $\angle BDC = \angle BAC$ or $\angle BDC = \pi - \angle BAC$. Since the planes through BC that are tangent at P_A and P_D to the insphere of $ABCD$ are symmetric about the plane that bisects the dihedral angle, we have $\angle BP_AC = \angle BP_DC$. But P_A lies inside triangle BCD, whence $\angle BDC < \angle BP_AC = \angle BP_DC = \pi - \angle BAC$. (The last equality is a consequence of P_D being the orthocentre.) It follows that $\angle BDC = \angle BAC$. Analogously we get $\angle CDA = \angle CBA$ and $\angle ADB = \angle ACB$.

Keeping in mind these pairs of equal angles, we now look at the net of our tetrahedron.



We deduce that circumcircles of BCD_1, CAD_2, ABD_3 pass through P_D. From $AD_2 = AD_3$ we get $\angle D_2P_DA = \angle AP_DD_3$ or $\angle D_2P_DA = \pi - \angle AP_DD_3$, with analogous possibilities for the pairs $\angle D_3P_DB$, $\angle BP_DD_1$ and $\angle D_1P_DC$, $\angle CP_DD_2$. Since at most one of these three pairs can form a straight angle (because D_1 cannot lie on D_2D_3), we may assume (without loss of generality), that $\angle D_3P_DB = \angle BP_DD_1$ and $\angle D_1P_DC = \angle CP_DD_2$. Now two sides, an angle, and circumradius of ΔBP_DD_3 equal the corresponding elements of ΔBP_DD_1, whence they are congruent. Similarly, triangles CP_DD_2 and CP_DD_1 are also congruent. This implies that $P_DD_3 = P_DD_1 = P_DD_2$ and, therefore, triangles AP_DD_2 and AP_DD_3 are congruent by SSS. So we have $\angle P_DAD_2 = \pi - \angle D_2CP_D = \pi - \angle P_DCD_1 = \angle P_DBD_1 = \angle D_3BP_D = \pi - \angle D_3AP_D$, which implies that points D_2, A, D_3 are collinear; furthermore, the equality $D_3A = AD_2$ implies that A is midpoint of D_2D_3. Analogously B, C are midpoints of D_3D_1 and D_1D_2, whence the faces of $ABCD$ are congruent.

To summarize the previous paragraph, the assumption that O coincides with the excentre opposite vertex D has led to the conclusion that the tetrahedron $ABCD$ is isosceles. But the automorphism group of an isosceles tetrahedron contains the three involutions that interchange the endpoints of opposite edges, whence by symmetry, O must also coincide with other excentres. This contradiction shows that O cannot coincide with the centre of an exsphere that is tangent to a face of $ABCD$. But in general, a tetrahedron will also have three exspheres that are tangent to the sides of the dihedral angles determined by pairs of opposite edges; thus, it remains to show that O cannot coincide with the centre of this type of exsphere.

To that end we assume, to the contrary, that the circumcentre coincides with the centre of the exsphere tangent to the interior side of the face planes through the edge AB, namely ABD and ABC, and to the exterior of the face planes through CD, namely CDA and CDB. As before, the point of tangency of the exsphere with plane DAB is the circumcenter of ΔDAB. Of course O_C lies inside the circumcircle. On the other hand, O_C is separated from A by BD (in the circle DAB on the exterior side of the plane CDB) and separated from B by AD (in the circle DAB on the exterior side of the plane CDA). This contradiction completes the proof.

References

[1] Nathan Altshiller-Court, *Modern Pure Solid Geometry*, Chelsea Publishing, Bronx, N.Y., 1964.

[2] Marcel Berger, *Geometry*, Vol. 2, Springer-Verlag, 1987.

[3] F. G.-M., *Exercices de géométrie—comprenant l'exposé des méthodes géométriques et 2000 questions résolues*, quatrième édition, Mame et Fils, Tours, 1907.

No other complete solutions have ever been received.

The failed solution I to Problem 478 claimed, incorrectly, that the proof is implicit in Altshiller-Court's solid-geometry text [1]. Klamkin and Liu refuted the claim [1987 : 151-152], thereby moving the problem to our unsolved problem list. Note an immediate consequence of the theorem: For a tetrahedron to be isosceles, it is sufficient (as well as necessary) that the circumcircles of the faces be congruent. This joins the list of better-known necessary and sufficient conditions, most of which can be found in [1, pages 103-112]; specifically, if a tetrahedron satisfies one of the following seven properties, it is isosceles and it satisfies them all.

1. *The opposite edges are congruent. (This is the standard definition.)*
2. *The faces are congruent.*
3. *The three face angles at each vertex sum to* 180°.
4. *The incentre coincides with the circumcentre.*
5. *The faces have the same perimeter.*
6. *The faces have the same area.*
7. *There exist exactly four exspheres.*

On the other hand, Problem 330 [1978: 263-264]—mentioned above in the statement of the current problem—asserts that the congruence of the incircles of the four faces does not *imply that the tetrahedron be isosceles.*

3611. [2011 : 48, 51] *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Given x, y, and z are positive integers such that

$$\frac{x(y+1)}{x-1}, \frac{y(z+1)}{y-1}, \text{ and, } \frac{z(x+1)}{z-1}$$

are positive integers. Find the smallest positive integer N such that $xyz \le N$.

Remark from the editor.

We make three preliminary observations: (1) If $(x,y,z) = (a,b,c)$ satisfies the conditions, then so also does (b,c,a) and (c,a,b); (2) each of the integers x,y,z must be not less than 2; (3) since $(x-1,x)$, $(y-1,y)$ and $(z-1,z)$ are coprime pairs, it follows that $x-1$ divides $y+1$, $y-1$ divides $z+1$ and $z-1$ divides $x+1$ for any valid triple (x,y,z). The first solution answers the question of the problem, while the second indicates how to obtain the set of all valid (x,y,z).

I. Composite of solutions by Joseph DiMuro, Biola University, La Mirada, CA, USA; and Oliver Geupel, Brühl, NRW, Germany.

We must have that $x-1 \le y+1$, $y-1 \le z+1$ and $z-1 \le x+1$. Equality cannot hold in all three cases, so we may suppose with no loss of generality that $2(z-1) \le x+1$.

Then $2(z-1) \le x+1 \le y+3 \le z+5$, whence $z \le 7$, $y \le 9$ and $x \le 11$, so that $xyz \le 7 \times 9 \times 11 = 693$. On the other hand, $(x,y,z) = (11,9,7)$ satisfies the conditions of the problem, so the smallest value of N is 693.

II. Solution by Prithwijit De, Homi Bhabha Centre for Science Education, Mumbai, India, modified by the editor.

Suppose, first, that two of x,y,z are equal, say $x = y$. Since $x-1$ must divide $y+1 = x+1 = (x-1)+2$, $x-1$ must divide 2.

If $x = 2$, then $z-1$ must divide 3, so that $z = 2$ or $z = 4$. If $x = 3$, then $z-1$ must divide 4, so that $z = 2$, $z = 3$ or $z = 5$. However, $(x,y,z) \ne (3,3,2)$ since $y(z+1) = 9$ is not a multiple of $y-1 = 2$. However $(x,y,z) = (2,2,2), (2,2,4), (3,3,3), (3,3,5)$ are all valid.

Suppose now that x,y,z are all distinct and that z is the largest. Since $z-1$ divides $x+1$, $x < z \le x+2$, so that $z = x+1$ or $z = x+2$. The first option is excluded since $z \ge 4$ and $z-1$ does not divide $z(x+1) = z^2 = (z-1)(z+1)+1$. Hence $z = x+2$.

If $x-1 = y+1$, then $y = x-2$, and $x-3$ divides $x+3 = (x-3)+6$. Therefore $x-3$ must be one of the numbers 1, 2, 3, 6. These lead to the valid triples $(4,2,6)$, $(5,3,7)$, $(6,4,8)$ and $(9,7,11)$.

On the other hand, let $x-1 < y+1$. Since $x-1 = z-3$ divides $y+1$ and $y+1 \le z$, then $z-3$ must have two multiples, namely $z-3$ and $y+1$, between $z-3$ and z inclusive. Hence $2(z-3) \le z$, so that $z \le 6$. Checking possible values of y leads to the triples $(2,2,4)$, $(3,3,5)$.

Therefore, up to a cyclic permutation, the valid triples are $(2,2,2)$, $(2,2,4)$, $(3,3,3)$, $(3,3,5)$, $(4,2,6)$, $(5,3,7)$, $(6,4,8)$, $(9,7,11)$. It follows that the minimum value of N is $9 \times 7 \times 11 = 693$.

Also solved by DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; GERHARDT HINKLE, Student, Central High School, Springfield, MO, USA; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; SHAUNDA SAWYER, California State University, Fresno, CA, USA; DIGBY SMITH, Mount Royal University, Calgary, AB;

ALBERT STADLER, Herrliberg, Switzerland; EDMUND SWYLAN, Riga, Latvia; and the proposer. One incomplete solution was received.

Some of the solutions were quite long and inefficient. Besides De, Bailey, Malikić, Sawyer, Stadler and Swylan gave the correct set of triples satisfying the conditions of the problem. Bailey, Bataille and Stadler solved the system of equations $y+1=m(x-1)$, $z+1=n(y-1)$, $x+1=k(z-1)$ to obtain

$$x = 1 + \frac{2[k(n+1)+1]}{mnk-1};$$
$$y = 1 + \frac{2[m(k+1)+1]}{mnk-1};$$
$$z = 1 + \frac{2[n(m+1)+1]}{mnk-1}.$$

Bailey showed that the minimum of m, n, k does not exceed 3 and worked through the cases. Stadler began with the observation that

$$x = \frac{1+2kn+2k+mnk}{mnk-1} \le \frac{5mnk+1}{mnk-1}$$
$$= 5 + \frac{6}{mnk-1} \le 11,$$

with equality if and only $(m,n,k) = (1,1,2)$. Since also y and z do not exceed 11, it is straightforward to go through the cases.

Bataille showed that if $mnk = 2$, then (x,y,z) must be $(11,9,7)$ in some cyclic order. When $mnk \ge 3$, he found that

$$\frac{x+y+z}{3} = \frac{3mnk+2(mn+nk+km)+2(m+n+k)+3}{3(mnk-1)}$$
$$\le \frac{(3+6+6)mnk+3}{3(mnk-1)} = 5 + \frac{6}{mnk-1} \le 8,$$

whence $xyz \le ((x+y+z)/3)^3 \le 512$.

3612. [2011 : 48, 51] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find all nonconstant polynomials P such that $P(\{x\}) = \{P(x)\}$, for all $x \in \mathbb{R}$, where $\{a\}$ denotes the fractional part of a.

Composite of submitted solutions, modified by the editor.

Since $\{P(x+1)\} = P(\{x+1\}) = P(\{x\}) = \{P(x)\}$, the polynomial $P(x+1) - P(x)$ is an integer for each real x. Since any polynomial assuming two distinct integer values assumes every intermediate value, $P(x+1) - P(x)$ is equal to the same integer a for each x. Since $P(x)$ is not constant, it can assume each of its values only finitely often, so that $a \ne 0$.

Let $Q(x) = P(x) - ax$. Then $Q(x+1) - Q(x) = 0$ so that $Q(x)$ is periodic with period 1, and so bounded. Hence $Q(x) = b$ for some constant b and all real x. Therefore $P(x) = ax + b$. Since $b = P(0) = \{P(0)\}$, we must have $0 \le b < 1$. Let $x = 1/|a|$. Then

$$a\left\{\frac{1}{|a|}\right\} + b = \{\pm 1 + b\} = b,$$

so that $a\{1/|a|\} = 0$ and $a = \pm 1$.

Suppose that $a = -1$ and $b < x < 1$. Then

$$-(x-b) = -x + b = -\{x\} + b = \{-x+b\} \ge 0,$$

a contradiction. Therefore $a = 1$. Let $x = 1 - b$. Then $0 < x \leq 1$ and

$$0 \leq b \leq \{1 - b\} + b = P(\{1 - b\}) = \{P(1 - b)\} = \{1\} = 0,$$

whence $b = 0$. Therefore the sole solution to the problem is $P(x) = x$.

Solved by ROY BARBARA, Lebanese University, Fanar, Lebanon; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; ALBERT STADLER, Herrliberg, Switzerland; and the proposer. There were 5 incorrect solutions received.

3613. [2011 : 112, 114] *Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

Solve the system of equations

$$\frac{x(y+1)}{x-1} = 7\,, \; \frac{y(z+1)}{y-1} = 5\,, \text{and} \; \frac{z(x+1)}{z-1} = 12,$$

where x, y, and z are positive integers.

I. Composite of essentially the same solution by Roy Barbara, Lebanese University, Fanar, Lebanon; Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; Kathleen E. Lewis, University of the Gambia, Brikama, Gambia; and Michael Parmenter, Memorial University of Newfoundland, St. John's, NL.

The only solution is $(x, y, z) = (7, 5, 3)$.

First note that $x, y, z \geq 2$. We have $x(y+1) = 7(x-1)$ so that $x \mid 7(x-1)$. Since x and $x-1$ are coprime, $x \mid 7$ so $x = 7$ follows. Similarly, from $y(z+1) = 5(y-1)$ we deduce that $y = 5$. Finally, from $z + 1 = \frac{5(y-1)}{y} = 4$ we obtain $z = 3$.

II. Similar solutions by Dionne Bailey, Elsie Campbell, and Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; and Titu Zvonaru, Cománeşti, Romania.

Since $y + 1 = \frac{7(x-1)}{x} = 7 - \frac{7}{x}$ we have $6 - y = \frac{7}{x}$ so $x(6-y) = 7$. Since $x > 1$ we must have $x = 7$ and $6 - y = 1$ so $y = 5$. Using either one of the other two equations, $z = 3$ follows.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; A. WIL EDIE, Missouri State University, Springfield, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; ALBERT STADLER, Herrliberg, Switzerland; STAN WAGON, Macalester College, St. Paul, MN, USA; MENG XIONG, California State University, Fresno, CA, USA; and the proposer.

Heuver, Wagon and Zvonaru solved the given equations in real numbers and found another solution given by $(x, y, z) = \left(\frac{32}{19}, \frac{59}{32}, \frac{76}{59}\right)$.

3614. [2011 : 112, 114] *Proposed by Neven Jurič, Zagreb, Croatia.*

Taking consecutive decimal digits of $\frac{1}{7}$ the set of points $A(1, 4)$, $B(4, 2)$, $C(2, 8)$,... in the plane is obtained. Prove that all these points belong to the same ellipse. Compute the area of the ellipse.

Composite of the almost identical solutions of Roy Barbara, Lebanese University, Fanar, Lebanon; and Missouri State University Problem Solving Group, Springfield, MO, USA.

Since $\frac{1}{7} = \overline{142857}$, there are only six points: $(1,4), (4,2), (2,8), (8,5), (5,7)$, and $(7,1)$. Note that they are symmetric about the centroid $\left(\frac{9}{2}, \frac{9}{2}\right)$. Translate the centroid to the origin, and in the resulting coordinate system these points become

$$\left(-\frac{7}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{5}{2}\right), \left(-\frac{5}{2}, \frac{7}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{2}\right), \left(\frac{5}{2}, -\frac{7}{2}\right).$$

It is a straightforward exercise to verify that these points all lie on the ellipse

$$19x^2 + 36xy + 41y^2 = 306.$$

Finally, the area of this ellipse is

$$\frac{2\pi}{\sqrt{4AC - B^2}} = \frac{306\pi}{\sqrt{455}} \approx 45.07,$$

where we used $A = \frac{19}{306}, B = \frac{36}{306}, C = \frac{41}{306}$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Because the given points are the vertices of a centrally symmetric hexagon (whose opposite sides are parallel), the points lie on a conic by Pascal's theorem. Because the hexagon is also convex, the conic must be an ellipse. The equation of this conic is required only to find the area it encloses. Other solvers used the conic in its original position where its equation is $19x^2 + 36xy + 41y^2 - 333x - 531y + 1638 = 0$. *According to the* MathWorld *web page, this ellipse is known as the* one-seventh ellipse*; the reference given there is to David Wells,* The Penguin Dictionary of Curious and Interesting Numbers *(Penguin Books, 1986). For those readers with access to* Zeitschrift MNU *(Mathematischer und Naturwissenschaftlicher Unterricht), Milan Koman provides a modest generalization (Variationen auf die Ein-Siebtel-Ellipse, MNU* **63***:4 (2010) 200-202). These references were kindly provided by Rudolf Fritsch (University of Munich).*

3615. [2011 : 112, 114] *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Prove that if $x, y, z \geq 0$ and $x + y + z = 1$, then

$$\frac{xy}{\sqrt{z + xy}} + \frac{yz}{\sqrt{x + yz}} + \frac{zx}{\sqrt{y + zx}} \;\leq\; \frac{1}{2}.$$

Similar solutions by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; George Apostolopoulos, Messolonghi, Greece; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; Titu Zvonaru, Comǎneşti, Romania; and the proposer.

Since $z+xy = (z+x)(z+y)$, we can use the Arithmetic-Geometric Means Inequality to obtain that

$$\frac{xy}{\sqrt{z + xy}} = \frac{xy}{\sqrt{(z + x)(z + y)}} \leq \frac{xy}{2}\left(\frac{1}{z + x} + \frac{1}{z + y}\right).$$

Combining this with similar inequalities for the other two terms shows that the left side does not exceed

$$\frac{xy}{2}\left(\frac{1}{z + x} + \frac{1}{z + y}\right) + \frac{yz}{2}\left(\frac{1}{x + y} + \frac{1}{x + z}\right) + \frac{zx}{2}\left(\frac{1}{y + x} + \frac{1}{y + z}\right) = \frac{1}{2}.$$

Equality occurs if and only if (x,y,z) is equal to $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$, $(\frac{1}{2},\frac{1}{2},0)$, $(\frac{1}{2},0,\frac{1}{2})$ or $(0,\frac{1}{2},\frac{1}{2})$.

An alternative approach begins with the equation $z+xy=(1-x)(1-y)$ and its analogues.

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and ALBERT STADLER, Herrliberg, Switzerland.

3616. *Proposed by Dinu Ovidiu Gabriel, Valcea, Romania.*

Compute

$$L = \lim_{n\to\infty} n^{2k}\left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k+1)}{n^k+1}\right],$$

where $k \in \mathbb{R}$.

Comment by the editor.

All solvers noted that when $k=0$ then, since $n^0=1$ for $n>0$ the limit was $\dfrac{\pi}{4} - \dfrac{\arctan(2)}{2}$. And when $k<0$ we have, where $K=-k>0$,

$$\begin{aligned}
&\lim_{n\to\infty} n^{2k}\left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k+1)}{n^k+1}\right]\\
&= \lim_{n\to\infty}\left[\frac{\arctan\left(\frac{1}{n^K}\right)}{n^K} - \frac{\arctan\left(\frac{1}{n^K}+1\right)}{n^K(n^K+1)}\right]\\
&= 0.
\end{aligned}$$

The rest of the solutions are for the more interesting case when $k>0$.

I. Solution by Missouri State University Problem Solving Group, Springfield, MO, USA.

If $k>0$, letting $N=n^k$ and using the fact that $\arctan x = \frac{\pi}{2} - \frac{1}{x} + O\left(\frac{1}{x^2}\right)$ for large x, the limit may be rewritten as

$$\begin{aligned}
&\lim_{N\to\infty} N^2\left[\frac{\arctan(N)}{N} - \frac{\arctan(N+1)}{N+1}\right]\\
&= \lim_{N\to\infty} N^2\left[\frac{\frac{\pi}{2} - \frac{1}{N} + O\left(\frac{1}{N^2}\right)}{N} - \frac{\frac{\pi}{2} - \frac{1}{N+1} + O\left(\frac{1}{N^2}\right)}{N+1}\right]\\
&= \lim_{N\to\infty} \frac{N^2}{N(N+1)}\left[\frac{\pi}{2} - \frac{N+1}{N} + \frac{N}{N+1} + O\left(\frac{1}{N}\right)\right]\\
&= 1\cdot\left[\frac{\pi}{2} - 1 + 1 + 0\right]\\
&= \frac{\pi}{2}.
\end{aligned}$$

II. Solution by Michel Bataille, Rouen, France.

Let $v_n = n^{2k}\left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k+1)}{n^k+1}\right]$.

Suppose $k>0$. Let $f(x) = \frac{\arctan x}{x}$. Then, $f'(x) = -\frac{\phi(x)}{x^2}$ where $\phi(x) = \arctan(x) - \frac{x}{x^2+1}$. Since $\phi'(x) = \frac{2x^2}{(x^2+1)^2} > 0$ for $x>0$, the function ϕ is increasing on $(0, \infty)$ and therefore $0 < \phi(x) < \frac{\pi}{2} = \lim_{x\to\infty} \phi(x)$ for positive x. Observing that

$$v_n = n^{2k} \int_{n^k}^{n^k+1} \frac{\phi(x)}{x^2} dx$$

we may write for all $n \geq 1$,

$$n^{2k}\phi(n^k) \int_{n^k}^{n^k+1} \frac{dx}{x^2} \leq v_n \leq n^{2k}\phi(n^k+1) \int_{n^k}^{n^k+1} \frac{dx}{x^2}$$

that is,

$$\phi(n^k)\frac{n^{2k}}{n^k(n^k+1)} \leq v_n \leq \phi(n^k+1)\frac{n^{2k}}{n^k(n^k+1)}.$$

Since $\lim_{n\to\infty} \frac{n^{2k}}{n^k(n^k+1)} = 1$ and $\lim_{n\to\infty} \phi(n^k) = \lim_{n\to\infty} \phi(n^k+1) = \frac{\pi}{2}$, we obtain $L = \frac{\pi}{2}$ by the squeeze principle.

III. Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA, expanded by the editor.

Note that

$$\begin{aligned}
&n^{2k}\left[\frac{\arctan(n^k)}{n^k} - \frac{\arctan(n^k+1)}{n^k+1}\right] \\
&\quad = n^k \arctan(n^k) - \frac{n^{2k}}{n^k+1}\arctan(n^k+1) \\
&\quad = n^k \arctan(n^k) - \frac{(n^k+1)^2 - 2n^k - 1}{n^k+1}\arctan(n^k+1) \\
&\quad = n^k \arctan(n^k) - (n^k+1)\arctan(n^k+1) + \frac{2n^k+1}{n^k+1}\arctan(n^k+1). \qquad (1)
\end{aligned}$$

Let $f(x) = -x\arctan x$. Then f is continuous and differentiable on $(0, \infty)$. For each n, we have, by applying the Mean Value Theorem to the interval $[n^k, n^k+1]$, that there exists a c_n with $f'(c_n) = n^k \arctan(n^k) - (n^k+1)\arctan(n^k+1)$. Thus, since

$$f'(x) = -\frac{x}{x^2+1} - \arctan x, \qquad (2)$$

then from (1) and (2) we get

$$\begin{aligned}
L &= \lim_{n\to\infty}\left(-\frac{c_n}{c_n^2+1} - \arctan c_n\right) + \lim_{n\to\infty}\frac{2n^k+1}{n^k+1}\arctan(n^k+1) \\
&= 0 - \frac{\pi}{2} + 2\cdot\frac{\pi}{2} = \frac{\pi}{2}.
\end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANASTASIOS KOTRONIS, Athens, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3617. [2011 : 112, 115] *Proposed by Michel Bataille, Rouen, France.*

Let r be a positive rational number. Show that if r^r is rational, then r is an integer.

Solution by Oliver Geupel, Brühl, NRW, Germany.

Let $a, b, c,$, and d be positive integers such that

$$r = \frac{a}{b},\ r^r = \frac{c}{d},\ gcd(a, b) = 1.$$

It follows that

$$\frac{a^a}{b^a} = r^a = r^{rb} = \frac{c^b}{d^b}.$$

We have to prove that $b = 1$. Assume the contrary. Then, b has a prime divisor p. Let $\varepsilon(n)$ denote the exact exponent of p in the integer n, that is, the non-negative integer α such that $p^\alpha \mid n$ and $p^{\alpha+1} \nmid n$. We obtain

$$a \cdot \varepsilon(b) + b \cdot \varepsilon(c) = \varepsilon(b^a c^b) = \varepsilon(a^a d^b) = a \cdot \varepsilon(a) + b \cdot \varepsilon(d) = b \cdot \varepsilon(d).$$

Hence, $a \cdot \varepsilon(b) = b \cdot (\varepsilon(d) - \varepsilon(c))$, from which we deduce that b divides the positive number $\varepsilon(b)$. Consequently, $b < p^b \le p^{\varepsilon(b)} \le b$. This is a contradiction, which completes the proof.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; KAYLIN EVERETT, California State University, Fresno, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3618. [2011 : 113, 115] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $\alpha > 3$ be a real number. Find the value of

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n}{(n+m)^\alpha}.$$

Composite of nearly identical solutions by Anastasios Kotronis, Athens, Greece; and Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy.

The original sum may be rewritten as

$$\begin{aligned}\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n}{(n+m)^\alpha} &= \sum_{q=2}^{\infty} \sum_{p=1}^{q-1} \frac{p}{q^\alpha} = \sum_{q=2}^{\infty} \frac{1}{2} \cdot \frac{q(q-1)}{q^\alpha} = \frac{1}{2}\left[\sum_{q=2}^{\infty} \frac{1}{q^{\alpha-2}} - \sum_{q=2}^{\infty} \frac{1}{q^{\alpha-1}}\right] \\ &= \frac{1}{2}[(\zeta(\alpha-2) - 1) - (\zeta(\alpha-1) - 1)] \\ &= \frac{1}{2}(\zeta(\alpha-2) - \zeta(\alpha-1) - 1),\end{aligned}$$

where $\zeta(z)$ is the Riemann zeta function.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; ALBERT STADLER, Herrliberg, Switzerland; STAN WAGON, Macalester College, St. Paul, MN, USA; and the proposer.

Stadler and AN-anduud Problem Solving Group used symmetry to write

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n}{(n+m)^{\alpha}}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{(n+m)^{\alpha}}=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(n+m)-m}{(n+m)^{\alpha}}=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)^{\alpha-1}},$$

from which they proceeded in a similar fashion to the featured solution. The proposer proved

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)^{p}}=\zeta(p-1)-\zeta(p),$$

from which he used a symmetry argument similar to Stadler and AN-anduud's to set up a situation where the lemma could be used. Perfetti also supplied a proof of the convergence of the double sum. Wagon pointed out that Mathematica produces the desired result.

3619. [2011 : 113, 115] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$(a^2b-c)(b^2c-a)(c^2a-b)\ \leq\ 4(ab+bc+ca-3a^2b^2c^2)\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

First we show that

$$a^2b+b^2c+c^2a+abc\leq 4. \qquad (1)$$

By the cyclicity of (1), we may without loss of generality suppose that either $a\leq b\leq c$ or $a\geq b\geq c$. In either case, the Rearrangement Inequality yields

$$a^2b+b^2c+c^2a+abc=a\cdot ab+b\cdot bc+c\cdot ac+b\cdot ac\leq a\cdot ab+b\cdot ac+b\cdot ac+c\cdot bc=b(a+c)^2.$$

By the AM-GM Inequality, we have

$$b(a+c)^2=\frac{1}{2}\cdot 2b(a+c)(a+c)\leq\frac{1}{2}\left(\frac{2b+(a+c)+(a+c)}{3}\right)^3=4,$$

which completes the proof of (1).

As a consequence of (1),

$$\sum_{\text{cyclic}}a^3b^2+\sum_{\text{cyclic}}(a^3bc+2a^2b^2c)=(a^2b+b^2c+c^2a+abc)(ab+bc+ca)\leq 4(ab+bc+ca).$$

Moreover, by the AM-GM Inequality, we have

$$1=\left(\frac{a+b+c}{3}\right)^3\geq abc\geq(abc)^3$$

and

$$12(abc)^2\leq 12(abc)^{11/6}\leq\sum_{\text{cyclic}}(a^4bc^2+a^3bc+2a^2b^2c).$$

By expanding the expressions in the desired inequality, we finally obtain

$$\begin{aligned}
&(a^2b-c)(b^2c-a)(c^2a-b)-4(ab+bc+ca-3a^2b^2c^2)\\
&=((abc)^3-abc)+12(abc)^2-\sum_{\text{cyclic}}a^4bc^2+\left(\sum_{\text{cyclic}}a^3b^2-4(ab+bc+ca)\right)\\
&\le((abc)^3-abc)+\left(12(abc)^2-\sum_{\text{cyclic}}(a^4bc^2+a^3bc+2a^2b^2c)\right)\\
&\le 0,
\end{aligned}$$

and we are done.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece(2 solutions with Maple); and the proposer.

3620. [2011 : 113, 115] *Proposed by John G. Heuver, Grande Prairie, AB.*

Let P be an interior point in tetrahedron $ABCD$ and let AP, BP, CP, and DP meet the corresponding opposite faces in A', B', C', and D'. Then

$$\begin{aligned}
\frac{AP}{PA'}\frac{BP}{PB'}\frac{CP}{PC'}\frac{DP}{PD'} \quad = \quad & 3+2\left(\frac{AP}{PA'}+\frac{BP}{PB'}+\frac{CP}{PC'}+\frac{DP}{PD'}\right)\\
&+\frac{AP}{PA'}\frac{BP}{PB'}+\frac{AP}{PA'}\frac{CP}{PC'}+\frac{AP}{PA'}\frac{DP}{PD'}\\
&+\frac{BP}{PB'}\frac{CP}{PC'}+\frac{BP}{PB'}\frac{DP}{PD'}+\frac{CP}{PC'}\frac{DP}{PD'}.
\end{aligned}$$

Solution by the proposer.

Let a, b, c, and d, respectively, be the volumes of the four tetrahedra $PBCD$, $PACD$, $PABD$, and $PABC$ that form a partition of $ABCD$. Without loss of generality we assume Volume$(ABCD)=1$. Because $ABCD$ and $PBCD$ share the same base ΔBCD while their heights are in the ratio $\frac{AA'}{PA'}$, we have

$$\frac{AA'}{PA'}=\frac{\text{Volume}(ABCD)}{\text{Volume}(PBCD)}=\frac{a+b+c+d}{a}=\frac{1}{a}.$$

Therefore

$$\frac{AP}{PA'}=\frac{AA'-PA'}{PA'}=\frac{1}{a}-1.$$

Similarly,

$$\frac{BP}{PB'}=\frac{1}{b}-1,\quad \frac{CP}{PC'}=\frac{1}{c}-1,\quad \text{and}\quad \frac{DP}{PD'}=\frac{1}{d}-1.$$

We seek the product, Π, of these four terms. We now have

$$\begin{aligned}
\Pi &= \frac{AA'}{PA'}\cdot\frac{BP}{PB'}\cdot\frac{CP}{PC'}\cdot\frac{DP}{PD'}\\
&=\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)\left(\frac{1}{d}-1\right)\\
&=\frac{1}{abcd}-\frac{1}{abc}-\frac{1}{abd}-\frac{1}{acd}-\frac{1}{bcd}+\frac{1}{ab}+\frac{1}{ac}+\frac{1}{ad}\\
&\qquad+\frac{1}{bc}+\frac{1}{bd}+\frac{1}{cd}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}-\frac{1}{d}+1.
\end{aligned}$$

Since $a+b+c+d=1$, we have

$$\frac{1}{abcd} = \frac{1}{abc} + \frac{1}{abd} + \frac{1}{acd} + \frac{1}{bcd}.$$

It follows that

$$\begin{aligned}\Pi &= \frac{1}{ab} + \frac{1}{ac} + \frac{1}{ad} + \frac{1}{bc} + \frac{1}{bd} + \frac{1}{cd} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d} + 1 \\ &= \sum_{\text{cyclic}} \left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right) - 6 + 2\sum_{\text{cyclic}} \left(\frac{1}{a} - 1\right) + 8 + 1 \\ &= 3 + 2\left(\frac{AP}{PA'} + \frac{BP}{PB'} + \frac{CP}{PC'} + \frac{DP}{PD'}\right) + \frac{AP}{PA'}\frac{BP}{PB'} + \frac{AP}{PA'}\frac{CP}{PC'} + \frac{AP}{PA'}\frac{DP}{PD'} \\ &\qquad + \frac{BP}{PB'}\frac{CP}{PC'} + \frac{BP}{PB'}\frac{DP}{PD'} + \frac{CP}{PC'}\frac{DP}{PD'}.\end{aligned}$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France;and OLIVER GEUPEL, Brühl, NRW, Germany.

Geupel observed that when distances are directed, the identity holds more generally for each point P that is not in a plane determined by a face of $ABCD$. Heuver found the problem in Nathan Altshiller Court's Modern Pure Solid Geometry *(Chelsea, 1964), page 141 #10. He proved the analogous theorem for a simplex in four dimensions (where the product of five quotients equals an expression with 26 summands) and conjectured that his approach would extend to simplices in n-dimensional Euclidean space.*