RECURRING CRUX CONFIGURATIONS 5

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Cyclic Orthodiagonal Quadrilaterals

A quadrilateral $ABCD$ is called *cyclic* if its vertices are arranged around its circumcircle in the same order as they appear in the name; it is *orthodiagonal* if $AC \perp BD$. Standard references such as [1, pages 136-139] list some of their familiar properties; other references that have been suggested by **Crux** readers are [1] and [3]. A familiar property that was generalized in a "Klamkin Quickie" is quick enough to reproduce here in full:

[2003 : 375, 377] For any four points A, B, C, D in Euclidean 3-space, AC and BD are orthogonal if and only if

$$AB^2 + CD^2 = BC^2 + DA^2. \tag{1}$$

Proof. Denote the vectors from A to B, C, and D by $\vec{B}, \vec{C}$, and $\vec{D}$, respectively. Then we have the identity

$$\begin{aligned} AB^2 + CD^2 - BC^2 - DA^2 &= \vec{B}^2 + (\vec{C} - \vec{D})^2 - (\vec{B} - \vec{C})^2 - \vec{D}^2 \\ &= 2\vec{C} \cdot (\vec{B} - \vec{D}). \end{aligned}$$

Thus $AB^2 + CD^2 = BC^2 + DA^2$ if and only if $\vec{C} \cdot (\vec{B} - \vec{D}) = 0$, which is true if and only if AC and BD are orthogonal.

Moreover, if these four points are the vertices of an orthodiagonal quadrilateral inscribed in a circle of radius R, and the diagonals intersect in the point P, we have

$$PA^2 + PB^2 + PC^2 + PD^2 \ = \ AB^2 + CD^2 \ = \ BC^2 + DA^2 \ = \ 8R^2. \tag{2}$$

Problem 3 of the 1991 British Mathematical Olympiad [1993 : 5; 1994 : 69-70] asked for a proof of (2) and inquired about its converse; a nonsquare rectangle serves to show that the equality

$$PA^2 + PB^2 + PC^2 + PD^2 = \frac{1}{2}\left(AB^2 + CD^2 + BC^2 + DA^2\right)$$

fails to imply that a cyclic quadrilateral be orthodiagonal.

An interesting result that solver Joe Howard found to be an easy consequence of equation (1) appeared in **Crux** as Problem 3402 [2009 : 42, 44; 2010 : 50-51] (proposed by Mihály Bencze): *If D and E are the midpoints of sides AB and AC of triangle ABC, then $CD \perp BE$ if and only if $5BC^2 = AC^2 + AB^2$.*

Among other familiar properties of the cyclic orthodiagonal quadrilateral $ABCD$ found in [1] we have

- (*Brahmagupta's theorem*) Any line through the intersection point P of the diagonals of an orthodiagonal quadrilateral that is perpendicular to a side bisects the opposite side.
- The midpoints of the four sides and the projections of P (where the diagonals intersect) onto the sides all lie on a circle whose centre is the midpoint between P and the circumcentre of $ABCD$.

You might recognize the second item from our previous configurations column: *The projections of the point P onto the sides of a cyclic orthodiagonal quadrilateral form the vertices of a bicentric quadrilateral.* This is Problem 2209 [1997 : 47; 1998 : 112-113], which comes with a proof, numerous references, and a converse: *If $QRST$ is a bicentric quadrilateral, then the lines that are perpendicular at $Q, R, S,$ and T to the lines from the incentre P form the sides of a cyclic orthodiagonal quadrilateral whose diagonals meet at P.*

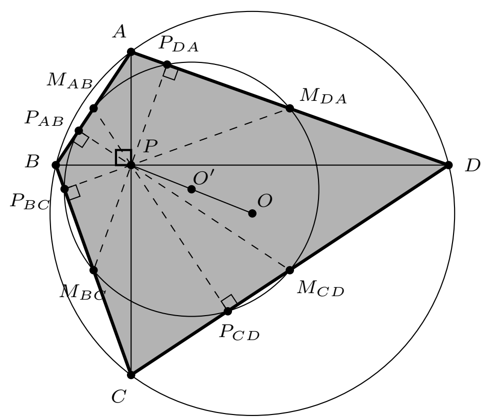

Figure 1: For the cyclic quadrilateral $ABCD$, $AC \perp BD$ implies that the midpoints of the four sides and the projections of P are cyclic; lines through P perpendicular to one side pass through the midpoint of the opposite side. Here P_{XY} and M_{XY} represent the projection of P and the midpoint of side XY, respectively.

Problem 1062 [1985 : 219; 1987 : 17-19] (Proposed by Murray S. Klamkin). If a convex quadrilateral is inscribed in a circle with centre O, then the distance from O to any side is half the length of the opposite side if and only if the diagonals are orthogonal.

Curiously, on page 138 of [1] Theorem 278 claims only that the orthodiagonal property of a cyclic quadrilateral implies the half-length property (and fails to mention the converse). The featured solution by Klamkin to part (b) of his problem proved, furthermore, that *given an oval which is centrosymmetric with centre O, should all convex orthodiagonal quadrilaterals inscribed in the oval have the property that the distance of any side from O is half the length of the opposite side, then that oval must be a circle.* It remains an open question whether an oval must be a circle if the half-length property of an inscribed convex quadrilateral always implies the orthodiagonal property.

Problem 1836 [1993 : 113; 1994 : 84-85] (proposed by Jisho Kotani and reworded here). If $ABCD$ is a cyclic quadrilateral, then the sum of the areas of the four crescent-shaped regions outside the circumcircle and inside the circles with diameters AB, BC, CD, DA equals the area of $ABCD$ if and only if its diagonals are orthogonal.

Problem 2338 [1998 : 234; 1999 : 243-245] (proposed by Toshio Seimiya, extended by Peter Y. Woo). When a convex quadrilateral is subdivided into four triangles by its two diagonals, then the incentres of the four triangles are the vertices of a cyclic orthodiagonal quadrilateral if and only if the initial quadrilateral has an incircle.

Problem 2978 [2004 : 429, 432; 2005 : 470-472] (proposed by Christopher J. Bradley). If $QRST$ is a cyclic quadrilateral whose adjacent interior angle bisectors intersect in the points A, B, C, and D, then $ABCD$ is a cyclic orthodiagonal quadrangle. Furthermore, the point where the diagonals AC and BD intersect is collinear with the circumcentres of the two quadrilaterals.

References

[1] Agnis Andžāns, On the Inscribed Orthodiagonal Quadrilaterals, *Mathematics and Informatics Quarterly*, **3**:1 (1993) 6-8.

[2] Nathan Altshiller Court, *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle*, 2nd ed. Barnes & Noble, 1965.

[3] Jordan Tabov, Simple Properties of Orthodiagonal Quadrilaterals, *Mathematics and Informatics Quarterly*, **1**:1 (1991) 1-5.