OC68. Find all integers x, y so that

$$x^3 + x^2 + x = y^2 + y\,.$$

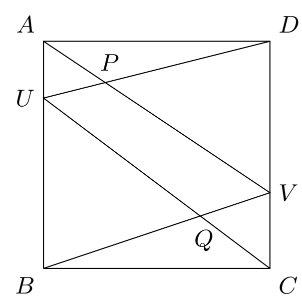
OC69. Let n be a positive integer and let $P(x, y) = x^n + xy + y^n$. Prove that we cannot find two non-constant polynomials $G(x, y)$ and $H(x, y)$ with real coefficients such that

$$P(x, y) = G(x, y) \cdot H(x, y)\,.$$

OC70. $\triangle ABC$ is a triangle such that $\angle C$ and $\angle B$ are acute. Let D be a variable point on BC such that $D \neq B, C$ and AD is not perpendicular to BC. Let d be the line passing through D and perpendicular to BC. Assume $d \cap AB = E$, $d \cap AC = F$. Let M, N, P be the incentres of $\triangle AEF$, $\triangle BDE$, $\triangle CDF$. Prove that A, M, N, P are concyclic if and only if d passes through the incentre of $\triangle ABC$.

OLYMPIAD SOLUTIONS

OC6. In the diagram, $ABCD$ is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral $PUQV$. (*Originally question # 3 from the 1992 Canadian Mathematical Olympiad.*)



Solved by Michel Bataille, Rouen, France; Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; and Titu Zvonaru, Comǎneşti, Romania. We give the solution of Bataille.

We will use $[\cdot]$ to denote the area. Let a be the length of the sides of the square. Since

$$[AVB] + [DUC] = \frac{a^2}{2} + \frac{a^2}{2} = [ABCD]\,, \qquad (1)$$

we get

$$[PUQV] = [APD] + [QBC]\,.$$

Let $u = AU$ and $v = CV$, and let P', Q' be the orthogonal projections of P onto AD, respectively Q onto BC. Then

$$\frac{AP'}{AD} = \frac{PP'}{DV} \Rightarrow AP' = \frac{AD}{DV}PP' = \frac{a}{a-v}PP'\,.$$

Similarly we get $DP' = \frac{a}{u}PP'$.

Then

$$a = AP' + DP' = \frac{a}{a-v}PP' + \frac{a}{u}PP'\,,$$

which yields

$$PP' = \frac{u(a-v)}{a+u-v}.$$

In a similar way, we obtain

$$QQ' = \frac{v(a-u)}{a+v-u}\,.$$

Since $[APD] = \frac{1}{2}a \cdot PP'$ and $[BQC] = \frac{1}{2}a \cdot QQ'$, (1) yields

$$[PUQV] = \frac{a}{2}\left(\frac{u(a-v)}{a+u-v} + \frac{v(a-u)}{a+v-u}\right).$$

Now, by AM-GM, $u(a-v) \le \frac{(a+u-v)^2}{4}$ and $v(a-u) \le \frac{(a+v-u)^2}{4}$, with equality if and only if $u+v=a$.

Thus

$$[PUQV] \le \frac{a}{8}(a+u-v+a+v-u) = \frac{a^2}{4}.$$

In conclusion, $[PUQV] \le \frac{a^2}{4}$ and the maximum value $\frac{a^2}{4}$ is attained exactly when $AU + VC = a$.

OC7. Let n be a natural number such that $n \ge 2$. Show that

$$\frac{1}{n+1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right).$$

(*Originally question # 3 from the 1998 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Manes.

Our inequality is equivalent to

$$n\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right).$$

Using the well known Catalan's inequality

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n} = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n},$$

we get

$$1+\frac{1}{3}+\cdots+\frac{1}{2n-1} = \left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)+\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right).$$

Thus the inequality to prove becomes

$$n\left(\frac{1}{2}+\frac{1}{4}+\dots\frac{1}{2n}\right)+n\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right) > (n+1)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right),$$

or equivalently

$$n\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right) > \left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right). \qquad (1)$$

Since $n \ge 2$, for all $1 \le j \le n$ we have $n+j < 2nj$ and hence:

$$\frac{n}{n+j} > \frac{1}{2j}.$$

Adding these inequalities yields (1) and completes the proof.

OC8. For each real number r let T_r be the transformation of the plane that takes the point (x, y) into the point $(2^r x, r2^r x + 2^r y)$. Let F be the family of all such transformations *i.e.* $F = \{T_r : r \in \mathbb{R}\}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in F.
(*Originally question # 2 from the 1983 Canadian Mathematical Olympiad.*)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Bataille.

It is readily checked that T_r is a bijective transformation with $T_r^{-1} = T_{-r}$. Thus, it suffices to find all curves $\mathcal{C}_f : y = f(x)$ such that $T_r(\mathcal{C}_f) \subset \mathcal{C}_f$ for all real r.
We show that the solutions are the curves $y = f_{n,p}(x)$ where $f_{n,p}$ is defined by

$$f_{n,p}(x) = \begin{cases} x(\log_2(x) + p) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ x(\log_2(|x|) + n) & \text{if } x < 0 \end{cases}$$

where n, p are real constants.

First, let $f_{n,p}$ be such a function, $M(x, f_{n,p}(x))$ be any point on $\mathcal{C}_{f_{n,p}}$ and r any real number. Then $T_r(M)$ has coordinates $x' = 2^r x, \quad y' = r2^r x + 2^r f_{n,p}(x)$.

If $x = 0$, then $y' = 2^r f_{n,p}(0) = 0 = f_{n,p}(x')$.

If $x < 0$, then

$$f_{n,p}(x') = 2^r x(\log_2(|2^r x|) + n) = 2^r x(r + \log_2(|x|) + n) = r2^r x + 2^r f_{n,p}(x) = y'$$

hence $y' = f_{n,p}(x')$ and $T_r(M)$ is on $\mathcal{C}_{f_{n,p}}$.

The case $x > 0$ is easily treated in a similar way. Thus,

$$T_r(\mathcal{C}_{f_{n,p}}) \subset \mathcal{C}_{f_{n,p}}\,.$$

Conversely, let f be such that $T_r(\mathcal{C}_f) \subset \mathcal{C}_f$ for all r. This means that for all r, x, we have

$$f(2^r x) = r2^r x + 2^r f(x)\,. \tag{1}$$

Taking $r = 1, x = 0$ in (1) yields $f(0) = 2f(0)$, hence we must have $f(0) = 0$.

Let $\alpha > 0$. Setting $r = \log_2(\alpha)$ and $x = 1$ in (1), we get

$$f(\alpha) = \alpha \log_2(\alpha) + \alpha f(1) = \alpha(\log_2(\alpha) + p)\,,$$

where $p = f(1)$.

Now setting $r = \log_2(\alpha)$ and $x = -1$ in (1), we get

$$f(-\alpha) = -\alpha \log_2(\alpha) + \alpha f(-1) = (-\alpha)(\log_2(|-\alpha|) + n)\,,$$

where $n = -f(-1)$. It follows that $f = f_{n,p}$.

OC9. A deck of $2n + 1$ cards consists of a joker and, for each number between 1 and n inclusive, two cards marked with that number. The $2n + 1$ cards are placed in a row, with the joker in the middle. For each k with $1 \le k \le n$, the two cards numbered k have exactly $k - 1$ cards between them. Determine all the values of n not exceeding 10 for which this arrangement is possible. For which values of n is it impossible?
(*Originally question # 5 from the 1992 Canadian Mathematical Olympiad.*)

Solution by Oliver Geupel, Brühl, NRW, Germany. No other solution was received.

We prove that such an arrangement is possible if and only if

$$n \in \{3, 4, 7, 8\} . \qquad (1)$$

Let's number the places from left to right $1, 2, 3, ..., 2n + 1$. The joker is placed on $n+1$. For each $1 \le k \le n$, let the two cards with number k be on position a_k and $a_k + k$. Then we have

$$\sum_{k=1}^{n} (a_k + a_k + k) = \left(\sum_{k=1}^{2n+1} k \right) - (n + 1) .$$

Thus

$$\sum_{k=1}^{n} a_k = \frac{3n(n + 1)}{4} .$$

Thus, either n or $n + 1$ is divisible by 4, which implies (1).

To complete the proof, we need to show that these n work. Indeed we have the following:

$$\begin{array}{lcc} n = 3 & : & 2, 3, 2, J, 3, 1, 1 \\ n = 4 & : & 2, 4, 2, 3, J, 4, 3, 1, 1 \\ n = 7 & : & 5, 3, 4, 7, 3, 5, 4, J, 6, 7, 1, 1, 6 \\ n = 8 & : & 6, 8, 5, 7, 1, 1, 6, 5, J, 8, 7, 4, 2, 3, 2, 4, 3 \end{array}$$

OC10. The number 1987 can be written as a three digit number xyz in some base b. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of x, y, z, b.
(*Originally question # 2 from the 1987 Canadian Mathematical Olympiad.*)

Solved by Florencio Cano Vargas, Inca, Spain; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comăneşti, Romania. We give the solution of Cano Vargas.

Let $xyz_{(b)} = 1987$.
Thus

$$xb^2 + yb + z = 1987 . \qquad (1)$$

Since x, y, z are digits in base b it follows immediately that $b > 10$ and $b^2 < 1987$.

From the problem we also have

$$x + y + z = 25. \qquad (2)$$

Subtracting (2) from (1) we get

$$(b - 1)\,(xb + x + y) = 1962 .$$

And hence

$$(b - 1) | 2 \cdot 3^2 \cdot 109 .$$

Since $b^2 < 1987$ we have $109 \nmid (b - 1)$ and hence $(b - 1)$ is a divisor of 18. Hence, using $b > 10$, we get $b - 1 = 18$. Thus $b = 19$.

We showed that $b = 19$ is the only possible solution. We see now that this works. Indeed, changing 1987 into base 19 we get

$$1987_{(10)} = 59E_{(19)} ,$$

where E is the digit eleven in base 19, and this satisfies the requirements of the problem.