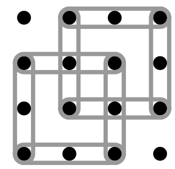
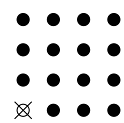
MAYHEM SOLUTIONS

M507. *Proposed by the Mayhem Staff.*

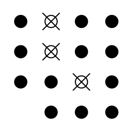


A 4 by 4 square grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form squares, two different 2 by 2 squares are shown in the diagram. What is the least number of pegs that must be removed so that **no** squares can be formed?

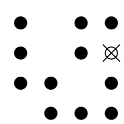
Solution by Juz'an Nari Haifa, student, SMPN 8, Yogyakarta, Indonesia.



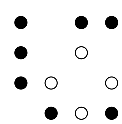
To find the least number of pegs that must be removed, we will take pegs away as necessary. We must consider squares of side length 3, 2, 1, $\sqrt{2}$ and $\sqrt{3}$. We will begin with a 3 by 3 square by eliminating the bottom left peg. Consequently, no 3 by 3 square can be formed.



Next we will remove any 2 by 2 squares. There are four different 2 by 2 squares, but one has already been eliminated. Since none of them share vertices, we need to remove 3 squares. Since we are looking for the smallest number of pegs, we will not remove any from the corners so we are left with the pegs shown in the diagram to the right.



At this point there is only one 1 by 1 square left so we will proceed by removing the bottom right peg of that square. Due to the pegs we chose to remove so far, neither of the two $\sqrt{5}$ by $\sqrt{5}$ squares can be formed.



This leaves us with only the $\sqrt{2}$ by $\sqrt{2}$ squares to consider. There is only one $\sqrt{2}$ by $\sqrt{2}$ left (marked with the open circles in the diagram) and we can remove any one of its pegs to remove the square. Thus we have removed 6 pegs and no more squares can be formed.

Note that we can apply the same method to obtain different answers, but the minimum number of pegs that need to be removed will always be 6.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MUHAMMAD LABIB IRFANUDDIN, student, SMP N 8 YOGYAKARTA, Indonesia; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; TAUPIEK DIDA PALLEVI, student, SMP N 8 YOGYAKARTA, Indonesia; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and ARIF SETYAWAN, student, SMP N 8 YOGYAKARTA, Indonesia. One incorrect solution was submitted.

M508. *Proposed by the Mayhem Staff.*

In 1770, Joseph Louis Lagrange proved that **every** non-negative integer can be expressed as the sum of the squares of four integers. For example $6 = 2^2+1^2+1^2+0^2$ and $27 = 5^2+1^2+1^2+0^2 = 4^2+3^2+1^2+1^2 = 3^2+3^2+3^2+0^2$ (in the theorem it is acceptable to use 0^2, or to use a square more than once). Notice that 27 had several different representations. Show that there is a number, not greater than 1 000 000 that can be represented as a sum of the squares of four **distinct non-negative** integers in more than 100 ways. (Note that rearrangements are not considered different, so $4^2 + 3^2 + 2^2 + 1^2 = 1^2 + 2^2 + 3^2 + 4^2$ are the same representation of 30.)

Solution by Florencio Cano Vargas, Inca, Spain.

We are going to count the number of combinations of squares of four integers, $(x_i^2, i = 1, 2, 3, 4)$, whose sum does not exceed 1 000 000. To get a conservative bound let us consider the case where all four integers are equal, i.e. $x_1 = x_2 = x_3 = x_4 \equiv x$. Then we have, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4x^2 \leq 10^6 \Rightarrow x \leq 500$. For any combination four non-negative integers not greater than 500, the sum of the squares of these four numbers is less than 1 000 000. Note that there are some combinations, like $1^2 + 2^2 + 3^2 + 600^2 = 360\ 014 < 1\ 000\ 000$, with some $x_i > 500$ but they are not included in our counting. Therefore we will limit ourselves to combinations $x_1^2 + x_2^2 + x_3^2 + x_4^2$ with $0 \leq x_i \leq 500$ and $x_i \neq x_j$ when $i \neq j$. Since each x_i can take 501 values, the number of combinations is given by $\binom{501}{4}$. [**Note**: If we had allowed combinations with repeated integers, we would have combinations with repetition, i.e. we would have had $\binom{504}{4}$ combinations to deal with].

We then have $\binom{501}{4} \approx 2.5937 \cdot 10^9$ combinations to assign to at most 1 000 000 numbers. This gives an average of

$$\binom{501}{4} \div 10^6 \approx 2593.7$$

combinations per integer, hence there is at least one integer not greater than 1 000 000 that can be represented as a sum of four distinct non-negative integer in 2594 ways by the pigeonhole principle.

Note: Following this method we can easily prove that we can find a number not greater than N that can be represented as a sum of squares of four integers in more than 100 ways, where N is the minimal solution of the inequality

$$\binom{\left\lfloor\sqrt{\frac{N}{4}}\right\rfloor + 1}{4} \div N \geq 100$$

which yields $N = 40\ 000$. It should be noted that one could find a better bound with a more refined counting method since we have discarded integers x_i which are larger than $\left\lfloor\frac{\sqrt{N}}{2}\right\rfloor$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA. One incorrect solution was submitted.

This problem was inspired by an example on page 22 of the book Numbers: A Very Short Introduction *by Peter M. Higgins, Oxford University Press, 2011. In the example the author showed that there is a number less than* 4 000 000 000 *that can be written as the sum of four different* cubes *in 10 distinct ways.*

M509. *Proposed by Titu Zvonaru, Comănești, Romania.*

Let ABC be a triangle with angles B and C acute. Let D be the foot of the altitude from vertex A. Let E be the point on AC such that $DE \perp AC$ and let M be the midpoint of DE. Show that if $AM \perp BE$, then $\triangle ABC$ is isosceles.

Solution by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania.

We will begin by setting up a set of coordinates such that the side BC is on the x-axis and the altitude AD is on the y-axis. Then we have $A(0,a)$, $B(-b,0)$, $C(c,0)$ and $D(0,0)$, where $a,b,c \in (0,\infty)$. The equation of line AC is $y = -\frac{a}{c}x + a$ and since $DE \perp AC$, the equation of the line DE is $y = \frac{c}{a}x$. Their point of intersection is $E\left(\frac{a^2c}{a^2+c^2}, \frac{ac^2}{a^2+c^2}\right)$.

The midpoint of DE is $M\left(\frac{a^2c}{2(a^2+c^2)}, \frac{ac^2}{2(a^2+c^2)}\right)$. The slopes of the lines AM and BE are respectively

$$m_{AM} = \frac{y_M - y_A}{x_M - x_A} = -\frac{2a^2+c^2}{ac}, \quad m_{BE} = \frac{y_E - y_B}{x_E - x_B} = \frac{ac^2}{a^2c + b(a^2+c^2)}.$$

These lines are perpendicular if and only if their slopes satisfy $m_{AM} \cdot m_{BE} = -1$. Thus

$$c(a^2+c^2) = b(a^2+c^2),$$

and since $a,b,c > 0$, we get $b = c$, hence it follows that triangle ABC is isosceles.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ADRIAN NACO, Polytechnic University of Tirana, Albania; RICARD PEIRÓ I ESTRUCH, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; MIHAÏ STOËNESCU, Bischwiller, France; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One incorrect solution was submitted.

M510. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

If $a,b,c \in \mathbb{C}$ such that $|a| = |b| = |c| = r > 0$ and $a+b+c \neq 0$, compute the value of the expression

$$\frac{|ab+bc+ca|}{|a+b+c|}$$

in terms of r.

Solution by Adrian Naco, Polytechnic University of Tirana, Albania.

Based on the properties of the complex numbers, we have that

$$\begin{aligned} 0 \neq |a+b+c|^2 &= (a+b+c)\overline{(a+b+c)} \\ &= a\overline{a} + b\overline{b} + c\overline{c} + (a\overline{b} + \overline{a}b) + (b\overline{c} + \overline{b}c) + (c\overline{a} + \overline{c}a) \\ &= |a|^2 + |b|^2 + |c|^2 + (a\overline{b} + \overline{a}b) + (b\overline{c} + \overline{b}c) + (c\overline{a} + \overline{c}a) \\ &= 3r^2 + (a\overline{b} + \overline{a}b) + (b\overline{c} + \overline{b}c) + (c\overline{a} + \overline{c}a). \end{aligned} \tag{1}$$

Furthermore,

$$\begin{aligned}|ab+bc+ca|^2 &= (ab+bc+ca)\overline{(ab+bc+ca)}\\ &= ab\overline{ab}+bc\overline{bc}+ca\overline{ca}\\ &\qquad +(ab\overline{bc}+\overline{ab}bc)+(bc\overline{ca}+\overline{bc}ca)+(ca\overline{ab}+\overline{ca}ab)\\ &= |a|^2|b|^2+|b|^2|c|^2+|c|^2|a|^2\\ &\qquad +|b|^2(a\overline{c}+\overline{a}c)+|c|^2(b\overline{a}+\overline{b}a)+|a|^2(c\overline{b}+\overline{c}b)\\ &= 3r^4+r^2(c\overline{a}+\overline{c}a)+r^2(b\overline{a}+\overline{b}a)+r^2(c\overline{b}+\overline{c}b)\\ &= r^2[3r^2+(c\overline{a}+\overline{c}a)+(b\overline{a}+\overline{b}a)+(c\overline{b}+\overline{c}b)]\,. \qquad (2)\end{aligned}$$

From (1) and (2) we get

$$\begin{aligned}|ab+bc+ca|^2 = r^2|a+b+c|^2 \ &\Rightarrow\ |ab+bc+ca| = r|a+b+c|\\ &\Rightarrow \frac{|ab+bc+ca|}{|a+b+c|} = r\,.\end{aligned}$$

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; IOAN VIOREL CODREANU, Secondary School student, Satulung, Maramureş, Romania; MARIAN DINCĂ, Bucharest, Romania; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANDHIKA GILANG, student, SMPN 8, Yogyakarta, Indonesia; THARIQ SURYA GUMELAR, student, SMPN 8, Yogyakarta, Indonesia; CORNELIU MĂNESCU-AVRAM, Transportation High School, Ploieşti, Romania; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRÓ I ESTRUCH, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HENRY RICARDO, Tappan, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and the proposer.

M511. *Proposed by Gili Rusak, student, Shaker High School, Latham, NY, USA.*

Pens come in boxes of 48 and 61. What is the smallest number of pens that can be bought in two ways if you must buy at least one box of each type?

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

Suppose we buy a boxes with 48 pens each and b boxes with 61 pens each, and in another way we buy c boxes with 48 pens each and d boxes with 61 pens each. Then we have $48a+61b$ in one way and $48c+61d$ in the other way. Since we want the number of pens to be the same in both ways, we have $48a+61b = 48c+61d$. If $a = c$ then $b = d$, in which case the two ways are the same. Since we set up two different ways of buying the pens, then we have $a \neq c$.

Since 48 and 61 are relatively prime, then from $48(a-c) = 61(b-d)$ it follows that 61 divides $48(a-c)$. Without loss of generality, we will suppose that $a > c$. It follows that the minimum possible value of $a-c$ is 61, and hence $48(c+61)+61b = 48c+61d$, or equivalently, $d = 48+b$.

Since we must buy at least one box of each type, the smallest possible value of b is $b = 1$ and then $d = 49$. Similarly, since the smallest possible value of c is

$c = 1$, it follows that $a = 62$. Therefore, the smallest number of pens that can be bought is $(48)(62) + (61)(1) = (48)(1) + (61)(49) = 3037$.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRÓ I ESTRUCH, IES "Abastos", Valencia, Spain; and the proposer.

M512. *Selected from a mathematics competition.*

A class of 20 students was given a three question quiz. Let x represent the number of students that answered the first question correctly. Similarly, let y and z represent the number of students that answered the second and the third questions correctly, respectively. If $x \geq y \geq z$ and $x + y + z \geq 40$, determine the smallest possible number of students who could have answered all three questions correctly in terms of x, y and z.
(*This questions was originally question 3c from the 2008 Galois Contest.*)

Solution by Florencio Cano Vargas, Inca, Spain.

The smallest number of students who have answered all three questions correctly corresponds to a situation where the number of students who have failed at least one question is maximum.

Taking into account that the number of students that failed question 1, 2 and 3 are $20 - x$, $20 - y$ and $20 - z$ respectively, then the number of students who answered at least one question incorrectly is at most:

$$\max(20, 20 - x + 20 - y + 20 - z) = \max(20, 60 - x - y - z)\,.$$

Since $x + y + z \geq 40$ then

$$60 - x - y - z = 60 - (x + y + z) \leq 60 - 40 = 20\,.$$

And therefore the number of students who have failed at least one question is at most $60 - (x + y + z) \leq 20$. The number of students that answered all three questions correctly is at least:

$$20 - (60 - x - y - z) = x + y + z - 40\,.$$

Furthermore, since $x + y + z \leq 60$ we have the boundaries for the answer as:

$$0 \leq x + y + z - 40 \leq 20\,.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.