Then

$$0 \equiv 3^n + 4^n \equiv 3^n(1 + a^n) \pmod{p} .$$

As 3 is invertible mod p, we get

$$a^n \equiv -1 \pmod{p} .$$

As n is odd, this implies

$$(-a)^n \equiv 1 \pmod{p} .$$

Let r be the order of $-a$ modulo p. Then $r \mid n$ and $r \mid p - 1$. As p is the smallest divisor of n, $r < p$ and $r \mid n$, it follows that $r = 1$. Hence

$$1 \equiv -a \equiv -4 \cdot 3^{-1} \pmod{p}$$

and hence

$$3 \equiv -4 \pmod{p} .$$

Thus $p \mid 7$ and hence $p = 7$.

Unsolved Crux Problem

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Recently, Chris Fisher published a list of unsolved problems from *Crux* [2010 : 545, 547]. Below is one of these unsolved problems. Note that the solution to part (a) has been published [2005 : 468-470] but (b) remains open.

2977. [2004 : 429, 432; 2005 : 468-470] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers, let $r = \sqrt[n]{a_1 a_2 \cdots a_n}$, and let

$$E_n = \frac{1}{a_1(1+a_2)} + \frac{1}{a_2(1+a_3)} + \cdots + \frac{1}{a_n(1+a_1)} - \frac{n}{r(1+r)} .$$

(a) Prove that $E_n \geq 0$ for

- (a_1) $n = 3$;
- (a_2) $n = 4$ and $r \leq 1$;
- (a_3) $n = 5$ and $\frac{1}{2} \leq r \leq 2$;
- (a_4) $n = 6$ and $r = 1$.

(b)★ Prove or disprove that $E_n \geq 0$ for

- (b_1) $n = 5$ and $r > 0$;
- (b_2) $n = 6$ and $r \leq 1$.