PROBLEM SOLVER'S TOOLKIT

No. 3

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The Problem Solver's Toolkit is a new feature in ***Crux Mathematicorum****. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

On a Triangle Inequality

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It is well known [4, p. 18] that, if A, B, C are the angles of a triangle, then

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}\,, \tag{1}$$

with equality if and only if $A = B = C$. (A proof is immediate from the concavity of the sine function on the interval $[0, \pi]$.) Vasić [1] generalized (1) to

$$x \sin A + y \sin B + z \sin C \leq \frac{\sqrt{3}}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)\,, \tag{2}$$

where $x, y, z > 0$. In [2], the author showed that (2) was a special case of the two-triangle inequality

$$4(xx' \sin A \sin A' + yy' \sin B \sin B' + zz' \sin C \sin C') \leq \left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)\left(\frac{y'z'}{x'} + \frac{z'x'}{y'} + \frac{x'y'}{z'}\right)\,. \tag{3}$$

Here we strengthen (2) to

$$x \sin A + y \sin B + z \sin C \leq \frac{1}{2}(yz + zx + xy)\sqrt{\frac{x + y + z}{xyz}}\,. \tag{4}$$

We start with the polar moment of inertia inequality [3],

$$(w_1 + w_2 + w_3)(w_1 R_1^2 + w_2 R_2^2 + w_3 R_3^2) \geq w_2 w_3 \alpha_1^2 + w_3 w_1 \alpha_2^2 + w_1 w_2 \alpha_3^2\,,$$

in which w_1, w_2, w_3 are arbitrary nonnegative numbers; $\alpha_1, \alpha_2, \alpha_3$ are the sides of a triangle $A_1A_2A_3$; and R_1, R_2, R_3 are the distances from an arbitrary point to the vertices of the triangle. Taking $R_1 = R_2 = R_3 = R$, the circumradius of the triangle, and using the power mean inequality,

$$\frac{w_2w_3\alpha_1^2 + w_3w_1\alpha_2^2 + w_1w_2\alpha_3^2}{w_2w_3 + w_3w_1 + w_1w_2} \geq \left\{\frac{w_2w_3\alpha_1 + w_3w_1\alpha_2 + w_1w_2\alpha_3}{w_2w_3 + w_3w_1 + w_1w_2}\right\}^2,$$

we obtain

$$R(w_1 + w_2 + w_3)\sqrt{w_2w_3 + w_3w_1 + w_1w_2} \geq w_2w_3\alpha_1 + w_3w_1\alpha_2 + w_1w_2\alpha_3\,. \qquad (5)$$

Now letting

$$w_1^2 = \frac{yz}{x}, \qquad w_2^2 = \frac{zx}{y}, \qquad w_3^2 = \frac{xy}{z},$$

and using $\alpha_i = 2R\sin A_i$ in (5), we obtain (4).

There is equality if and only if

$$\alpha_1 = \alpha_2 = \alpha_3$$

and the centroid of the weights w_1, w_2, w_3 at the respective vertices of the triangle coincides with the circumcentre. This entails that

$$w_1 = w_2 = w_3\,,$$

or, equivalently, that

$$x = y = z\,.$$

We have therefore shown that equality holds in (4) if and only if the triangle is equilateral.

Finally, to show that (4) is stronger than (2), we must establish that

$$\sqrt{3}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right) \geq (yz + zx + xy)\sqrt{\frac{x+y+z}{xyz}}$$

or, equivalently, that

$$3(y^2z^2 + z^2x^2 + x^2y^2)^2 \geq xyz(x+y+z)(yz+zx+xy)^2\,.$$

Letting $x = \frac{1}{u}$, $y = \frac{1}{v}$, and $z = \frac{1}{w}$ shows that this is equivalent to

$$3(u^2 + v^2 + w^2)^2 \geq (u+v+w)^2(vw + wu + uv)\,.$$

Since

$$\left(\frac{u^2 + v^2 + w^2}{3}\right)^2 \geq \left(\frac{u+v+w}{3}\right)^4$$

by the power mean inequality, it suffices finally to show that

$$\left(\frac{u+v+w}{3}\right)^2 \geq \frac{vw + wu + uv}{3}\,,$$

and this is equivalent to

$$(v-w)^2 + (w-u)^2 + (u-v)^2 \geq 0\,.$$

References

[1] P. M. Vasić, "Some Inequalities for the Triangle", *Publ. Eletrotehn. Fak. Univ. Beogradu, Ser. Mat. Fiz.*, No. 274 - No. 301 (1964) 121-126.

[2] M. S. Klamkin, "An Identity for Simplexes and Related Inequalties", *Simon Stevin*, 48 (1974-1975) 57-64.

[3] M. S. Klamkin, "Geometric Inequalities via the Polar Moment of Inertia", *Mathematics Magazine*, 48 (1975) 44-46.

[4] O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publl., Groningen, 1969.

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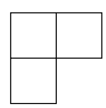
ATOM

One square is deleted from a square "checkerboard" with 2^{2n} squares. Show that the remaining $2^{2n}-1$ squares can always be tiled with shapes of the form



which cover three squares.

This appears as problem 26 from *Mathematical Olympiads' Correspondence Program (1995-96)* by Edward J. Barbeau which is Volume I of the Canadian Mathematical Society's booklet series **ATOM**.

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