SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3691. [2011 : 541, 543] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab} \;\leq\; 1\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Since for nonnegative real numbers such that $a+b+c=3$ we have

$$a^2b+b^2c+c^2a+abc\leq 4 \tag{1}$$

(See the lemma in the solution of Problem 3549 [2011:253].), then

$$\begin{aligned}
&4\left(\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab}-1\right)\\
&\qquad= a^2b\left(\frac{bc}{4-bc}+1\right)+b^2c\left(\frac{ca}{4-ca}+1\right)+c^2a\left(\frac{ab}{4-ab}+1\right)-4\\
&\qquad\leq \frac{a^2b^2c}{4-bc}+\frac{b^2c^2a}{4-ca}+\frac{c^2a^2b}{4-ab}-abc.
\end{aligned}$$

So it suffices to prove that

$$\frac{ab}{4-bc}+\frac{bc}{4-ca}+\frac{ca}{4-ab}-1\leq 0.$$

Clearing denominators, it becomes

$$\begin{aligned}
&32(ab+bc+ca)+abc(a^2b+b^2c+c^2a+abc)\\
&\qquad\qquad -64-8(a+b+c)abc-4(a^2b^2+b^2c^2+c^2a^2)\leq 0.
\end{aligned}$$

After applying inequality (1) and homogenizing, the inequality can be written in the form

$$\begin{aligned}
&\frac{32}{9}(a+b+c)^2(ab+bc+ca)+\frac{4}{3}(a+b+c)abc\\
&\qquad -\frac{64}{81}(a+b+c)^4-8(a+b+c)abc-4(a^2b^2+b^2c^2+c^2a^2)\leq 0.
\end{aligned}$$

Clearing denominators again, expanding and adopting the notation

$$[\alpha,\beta,\gamma]=\sum_{sym}a^{\alpha}b^{\beta}c^{\gamma},$$

it becomes $16([3,1,0]-[4,0,0])+33([2,1,1]-[2,2,0]) \leq 0$, which is true by Muirhead's theorem. This completes the proof.

Notice that the equality holds for $a=b=c=1$, or $a=2, b=1, c=0$, or any permutations of these values.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

The proposer posted this problem in 2008, on the Mathlinks forum, see `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=189307`. *Also, you can find a nice solution by Vo Quoc Ba Can on* `http://canhang2007.wordpress.com/2009/11/02/inequality-37-p-k-hung`.

3692. [2011 : 541, 543] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

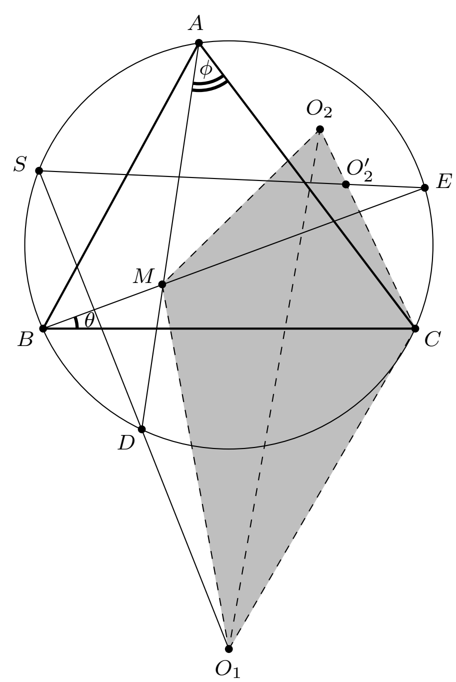
For an arbitrary point M in the plane of triangle ABC define D, E, and F to be the second points where the circumcircle meets the lines AM, BM, and CM, respectively. If O_1, O_2, and O_3 are the respective centres of the circles BCM, CAM, and ABM, prove that DO_1, EO_2, and FO_3 are concurrent.

Identical solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina and Salem Malikić, student, Simon Fraser University, Burnaby, BC, with the notation modified by the editor.

Editor's comment. To avoid the need for numerous special cases, we shall make use of directed angles: the symbol $\angle PQR$ will represent the angle through which the line QP must be rotated about Q to coincide with QR, where $0^\circ \leq \angle PQR < 180^\circ$. Properties of directed angles are discussed in Roger Johnson's *Advanced Euclidean Geometry*, Dover reprint (1960).

Our goal is to prove that not only are DO_1, EO_2, FO_3 concurrent, but their common point lies on the circumcircle of ΔABC. Denote by S the second point where the line DO_1 intersects the circumcircle, and let $\phi = \angle MAC = \angle DAC$ and $\theta = \angle CBM = \angle CBE$. Because A, C, D, E and S are all on the same circle we have

$$\begin{aligned}\angle O_1SE &= \angle DSE \\ &= \angle DSC + \angle CSE \\ &= \angle DAC + \angle CBE \\ &= \phi + \theta. \qquad (1)\end{aligned}$$



Since O_1 and O_2 are circumcentres of triangles MBC and MAC, respectively, we have

$$\angle MO_2C = 2\angle MAC = 2\phi$$

and

$$\angle CO_1M = 2\angle CBM = 2\theta.$$

Moreover, the radii $O_2M = O_2C$ and $O_1M = O_1C$, so that the quadrilateral MO_2CO_1 is a kite, implying that

$$\angle O_1O_2C = \frac{1}{2}\angle MO_2C = \phi \text{ and } \angle CO_1O_2 = \frac{1}{2}\angle CO_1M = \theta. \tag{2}$$

Consequently, in ΔO_1CO_2 we have

$$\angle O_1CO_2 = \phi + \theta. \tag{3}$$

We denote by O_2' the point where ES intersects CO_2 and will prove that in fact, $O_2' = O_2$. From (1) and (3) we have

$$\angle O_1CO_2' = \angle O_1CO_2 = \phi + \theta = \angle O_1SE = \angle O_1SO_2'.$$

Consequently the points O_1, C, O_2', S are concyclic, whence (with the help of (2))

$$\angle O_1O_2'C = \angle O_1SC = \angle DSC = \angle DAC = \phi = \angle O_1O_2C.$$

Because there is exactly one line through O_1, namely O_1O_2, that makes a *directed* angle of ϕ with the line CO_2, we conclude that $O_2' = O_2$; that is, EO_2 passes through S, as claimed. Interchanging the roles of B and C, one shows similarly that FO_3 also passes through S.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France (2 solutions); and the proposer.

Apostolopoulos provided a neat computation, using complex numbers to show that the point where DO_1 intersects the circumcircle is represented by an expression that is symmetric in A, B, and C.

3693. [2011 : 541, 543] *Proposed by Michel Bataille, Rouen, France.*

Given $k \in \left(\frac{1}{4}, 0\right)$, let $\{a_n\}_{n=0}^{\infty}$ be the sequence defined by $a_0 = 2$, $a_1 = 1$ and the recursion $a_{n+2} = a_{n+1} + ka_n$. Evaluate

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2}\,.$$

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Using roots of the characteristic equation, $x^2 - x - k = 0$, of the given recursion, we have

$$a_n = \left(\frac{1+\sqrt{1+4k}}{2}\right)^n + \left(\frac{1-\sqrt{1+4k}}{2}\right)^n, \qquad k \in (-1/4, 0)\,.$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_n}{n^2} &= \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\left(\frac{1+\sqrt{1+4k}}{2}\right)^n + \left(\frac{1-\sqrt{1+4k}}{2}\right)^n\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{n^2}x^n + \sum_{n=1}^{\infty}\frac{1}{n^2}(1-x)^n := S(x),\end{aligned}$$

where $x = \frac{1+\sqrt{1+4k}}{2}$. Since

$$\sum_{n=1}^{\infty}\frac{1}{n}x^n = \sum_{n=1}^{\infty}\int_0^x t^{n-1}dt = \int_0^x \sum_{n=1}^{\infty} t^{n-1}dt = \int_0^x \frac{1}{1-t}dt = -\ln(1-x)$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n}(1-x)^n = -\ln x,$$

then

$$\begin{aligned}S'(x) &= \sum_{n=1}^{\infty}\frac{x^{n-1}}{n} + \sum_{n=1}^{\infty}\frac{(1-x)^{n-1}}{n}\\ &= -\frac{1}{n}\ln(1-x) + \frac{1}{1-x}\ln x\\ &= -(\ln x\cdot\ln(1-x))'.\end{aligned}$$

Hence

$$S(x) = -\ln x\cdot\ln(1-x) + C.$$

Since $\lim_{x\to 0^+}\ln x\cdot\ln(1-x) = 0$ and $S(0) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, then $C = \frac{\pi^2}{6}$, and we have the final result

$$\sum_{n=1}^{\infty}\frac{a_n}{n^2} = \frac{\pi^2}{6} - \ln\frac{1+\sqrt{1+4k}}{2}\cdot\ln\frac{1-\sqrt{1+4k}}{2}.$$

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RADOUAN BOUKHARFANE, Polytechnique de Montréal, Montréal, PQ; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANASTASIOS KOTRONIS, Athens, Greece; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; M. A. PRASAD, India; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and YANPING XIA, Southeast Missouri State University, Cape Girardeau, MO, USA; and the proposer.

3694. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x, y, and z be nonnegative real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$\sqrt{1 - \left(\frac{x+y}{2}\right)^2} + \sqrt{1 - \left(\frac{y+z}{2}\right)^2} + \sqrt{1 - \left(\frac{x+z}{2}\right)^2} \ \geq \ \sqrt{6}\,.$$

[*Editor's note*: Both AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and Albert Stadler, Herrliberg, Switzerland, pointed out that the same problem by the same proposer had appeared as Example 1 of the article "Square it" published in Vol. 12, No. 5 (2008) of *Mathematical Excalibur* (`http://www.math.ust.hk/excalibur/v12_n5.pdf`). The solution given there, which used the Cauchy-Schwarz Inequality, is similar to most of the submitted solutions we received.]

Solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; M. A. PRASAD, India; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

3695. [2011 : 541, 544] *Proposed by Michel Bataille, Rouen, France.*

Let a be a positive real number. Find all strictly monotone functions $f : (0, \infty) \to (0, \infty)$ such that

$$(x + a)f(x + y) = af(yf(x))$$

for all positive x,y.

I. Solution by Oliver Geupel, Brühl, NRW, Germany.

It is straightforward to check that $f(x) = \frac{a}{x+a}$ is a solution. We prove that it is unique.

Suppose that f meets the requirements of the problem. Since f is monotone, it has a limit L (possibly infinite) from the right at 0. For each $x > 0$, we have that

$$\lim_{y\to 0^+} f(x + y) = \frac{a}{x + a} \lim_{y\to 0^+} f(yf(x)) = \frac{a}{x + a} L.$$

Thus the limit L must be a positive real number. Since a monotone function has at most countably many discontinuities, $f(x) = (aL)/(x + a)$ on a dense subset of $(0, \infty)$ and so $f(x)$ must be decreasing.

For any $x > 0$ and each $\epsilon > 0$, we can choose positive numbers x_1 and x_2 such that $x - \epsilon < x_1 < x < x_2 < x + \epsilon$ and

$$\frac{a}{x_1 + a} \cdot L = f(x_1) \geq f(x) \geq f(x_2) = \frac{a}{x_2 + a} \cdot L.$$

Therefore, we must have that

$$f(x) = \frac{aL}{x+a}$$

for all $x > 0$. Substituting this expression into the functional equation reveals that L must be 1. Note that we did not require the monotonicity to be strict.

II. Solution by M. A. Prasad, India.

Define $g(x) = f(ax)$ for $x > 0$. We get that

$$(x+1)g(x+y) = g(yg(x))$$

for $x, y > 0$. For sufficiently large values of y, we may write

$$\begin{aligned}(x+1)g(x+y) = g(1+(yg(x)-1)) &= \frac{1}{2}g((yg(x)-1)g(1)) \\ &= \frac{1}{2}g\left(\frac{(yg(x)-1)g(1)}{g(2x+1)}g(2x+1)\right) \\ &= \frac{1}{2}(2x+2)g\left(2x+1+\frac{yg(x)-1}{g(2x+1)}g(1)\right).\end{aligned}$$

Since the function is strictly monotone, it is one-one, so that

$$x+y = 2x+1+\frac{yg(x)-1}{g(2x+1)}g(1).$$

For each x, this holds for infinitely many values of y, so that, equating terms independent of y and coefficients of y, we get that

$$x = 2x+1-\frac{g(1)}{g(2x+1)} \Leftrightarrow (x+1)g(2x+1) = g(1).$$

and

$$g(2x+1) = g(x)g(1).$$

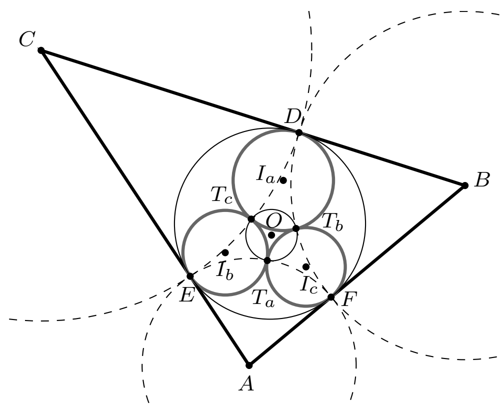
Eliminating $g(2x+1)$ from these equations yields that $(x+1)g(x) = 1$, from which $f(x)$ can be obtained.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and the proposer.

3696. [2011 : 541, 544] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let the incircle of triangle ABC touch the sides BC at D, CA at E, and AB at F. Construct by ruler and compass the three mutually tangent circles that are internally tangent to the incircle, one at D, one at E, and one at F.

Solution by George Apostolopoulos, Messolonghi, Greece.



We will use notation $\odot A$ to represent a circle with centre at A.
Let $\odot A$, $\odot B$, and $\odot C$ be the circles with radii AE, BF, and CD, respectively. First, we will construct a small circle $\odot O$, externally tangent to all of $\odot A$, $\odot B$, and $\odot C$.
(This is not a difficult but rather cumbersome construction and it can be found for example on the website:
`http://oz.nthu.edu.tw/~g9721504/soddycircles.html` *.)*
If T_a, T_b, T_c denote the corresponding tangent points, as on the diagram, then the circle $\odot I_c$ inscribed in $\triangle ABO$ is the circumcircle of $\triangle FT_aT_b$. Similarly, let $\odot I_a$ and $\odot I_b$ be the inscribed circles in $\triangle BCO$ and $\triangle CAO$, respectively.
We claim that the circles $\odot I_a$, $\odot I_b$, $\odot I_c$ satisfy conditions of the problem.
Firstly, they are tangent to the lines AB, BC, and CA at points E, D, and F, respectively, which means that they are tangent internally to the incircle of $\triangle ABC$ at the corresponding points E, D, and F.
Secondly, $\odot I_a$ and $\odot I_b$ are both tangent to the line CO at the common point T_c, hence they are externally tangent to each other. Since similar conclusion refers to the remaining pairs of circles, the claim holds.

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

3697. [2011 : 542, 544] *Proposed by Michel Bataille, Rouen, France.*

For positive integer n, prove that

$$\left(\tan\frac{\pi}{7}\right)^{6n}+\left(\tan\frac{2\pi}{7}\right)^{6n}+\left(\tan\frac{3\pi}{7}\right)^{6n}$$

is an integer and find the highest power of 7 dividing this integer.

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon, expanded slightly by the editor.

We prove more generally that $f(n)=\left(\tan\frac{\pi}{7}\right)^{2n}+\left(\tan\frac{2\pi}{7}\right)^{2n}+\left(\tan\frac{3\pi}{7}\right)^{2n}$ is an integer for all $n\in\mathbb{N}$.

Let $a=\left(\tan\frac{\pi}{7}\right)^2$, $b=\left(\tan\frac{2\pi}{7}\right)^2$ and $c=\left(\tan\frac{3\pi}{7}\right)^2$. Then $f(n)=a^n+b^n+c^n$.

It is well known (see [1]) that $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$, and $\tan\frac{3\pi}{7}$ are zeros of the polynomial $x^6 - 21x^4 + 35x^2 - 7$. Therefore a, b, and c are three (distinct) roots of $x^3 - 21x^2 + 35x - 7$.

It follows that $a + b + c = 21$, $ab + bc + ca = 35$ and $abc = 7$. Hence

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 21^2 - 70 = 371,$$

and

$$a^3+b^3+c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ca+3abc = 21(371-35)+21 = 7077.$$

Since

$$\begin{aligned}a^{n+3} + b^{n+3} + c^{n+3} &= (a + b + c)(a^{n+2} + b^{n+2} + c^{n+2})\\ &\quad - (ab + bc + ca)(a^{n+1} + b^{n+1} + c^{n+1}) + abc(a^n + b^n + c^n)\end{aligned}$$

we get

$$f(n + 3) = 21f(n + 2) - 35f(n + 1) + 7f(n) \tag{1}$$

and since $f(1) = 21$, $f(2) = 371$, and $f(3) = 7077$ are all integers, it follows from (1) that $f(n)$ is an integer for all $n \in \mathbb{N}$.

We now show that $7^n \parallel f(3n)$: that is, the highest power of 7 dividing $f(3n)$ is n.

Since $f(1)$, $f(2)$, and $f(3)$ are all multiples of 7, we have $7 \mid f(k)$ for $k = 1$, 2, and 3. Suppose that $7^k \mid f(3k - 2)$, $7^k \mid f(3k - 1)$ and $7^k \mid f(3k)$ for all $k = 1, 2, \ldots, n$ for some $n \geq 1$.

Then replacing n in (1) by $3n - 2$, $3n - 1$, and $3n$, respectively, we get

$$f(3n + 1) = 21f(3n) - 35f(3n - 1) + 7f(3n - 2) \tag{2}$$
$$f(3n + 2) = 21f(3n + 1) - 35f(3n) + 7f(3n - 1) \tag{3}$$
$$f(3n + 3) = 21f(3n + 2) - 35f(3n + 1) + 7f(3n). \tag{4}$$

Using (2) – (4) we obtain successively that $7^{n+1} \mid f(3n+1)$, $7^{n+1} \mid f(3n+2)$, and $7^{n+1} \mid f(3n + 3)$. That is, $7^{n+1} \mid f(3(n + 1))$. Hence by induction we conclude that $7^n \mid f(3n)$ for all $n \in \mathbb{N}$.

To complete the proof, it remains to show that $7^{n+1} \nmid f(3n)$. To this end, we use induction to prove that

$$\frac{f(3n)}{7^n} \equiv 3 \pmod 7\,. \tag{5}$$

Since $f(3) = 7077 = 7 \times 1011$ and $1011 \equiv 3 \pmod 7$, (5) is true for $n = 1$.

Suppose (5) holds for some $n \geq 1$. We let $f(3n) = 7^n q$ where $q \equiv 3 \pmod 7$. We also set $f(3n+1) = 7^{n+1}r$, $f(3n+2) = 7^{n+1}s$, and $f(3n+3) = 7^{n+1}t$, where r, s, and t are integers. Then by (4) we obtain $7^{n+1}t = 21(7^{n+1}s)-35(7^{n+1}r)+7(7^n q)$ which, upon dividing by 7^{n+1}, yields $t = 7(3s) - 7(5r) + q \equiv q \equiv 3 \pmod 7$. That is $\frac{f(3n+3)}{7^{n+1}} \equiv 3 \pmod 7$ and the induction is complete.

References

[1] mathworld.wolfram.com/TrigonometryAnglesPi7.html.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

3698. [2011 : 542, 544] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find the value of

$$\lim_{n\to\infty}\int_0^1 \sqrt[n]{1+n^n x^{n^2}}\,dx\,.$$

I. Solution by Albert Stadler, Herrliberg, Switzerland.

Let

$$I_n=\int_0^{1/\sqrt[n]{n}} \sqrt[n]{1+n^n x^{n^2}}dx,\quad\text{and}\quad J_n=\int_{1/\sqrt[n]{n}}^1 \sqrt[n]{1+n^n x^{n^2}}dx.$$

Then

$$\frac{1}{\sqrt[n]{n}}=\int_0^{1/\sqrt[n]{n}} dx\le I_n\le \int_0^{1/\sqrt[n]{n}} \sqrt[n]{2}dx\le \sqrt[n]{2},$$

and

$$\int_{1/\sqrt[n]{n}}^1 \sqrt[n]{n^n x^{n^2}}dx\le J_n\le \int_{1/\sqrt[n]{n}}^1 \sqrt[n]{2n^n x^{n^2}}dx.$$

Since

$$\int_{1/\sqrt[n]{n}}^1 \sqrt[n]{n^n x^{n^2}}dx=n\int_{1/\sqrt[n]{n}}^1 x^n dx=\frac{1}{n+1}\left(n-n^{-1/n}\right),$$

we find that $\lim_{n\to\infty} I_n=\lim_{n\to\infty} J_n=1$, so that the required limit is equal to 2.

II. Solution by George Apostolopoulos, Messolonghi, Greece; Moubinool Omarjee, Lycée Henri IV, Paris, France; and M. A. Prasad, India(independently).

Since $\sqrt[n]{1+n^n x^{n^2}}<1+nx^n$ for $0<x<1$. Then

$$\int_0^1 \sqrt[n]{1+n^n x^{n+1}}dx<\int_0^1 (1+nx^n)dx=1+\frac{n}{n+1}.$$

On the other hand, when $0<c<1$, we have that

$$\begin{aligned}\int_0^1 \sqrt[n]{1+n^n x^{n^2}}dx&=\int_0^c \sqrt[n]{1+n^n x^{n^2}}dx+\int_c^1 \sqrt[n]{1+n^n x^{n^2}}dx\\&>\int_0^c 1dx+\int_c^1 nx^n dx=c+\frac{n}{n+1}(1-c^{n+1}).\end{aligned}$$

Therefore the required limit is not less than $c+1$ for each $c \in (0,1)$. It follows that the limit is 2.

Also solved by PAUL BRACKEN, University of Texas, Edinburg, TX, USA; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. In the second solution, Omarjee avoided the extra parameter by taking the specific value $c = n^{-1/n}$. Perfetti and the proposer made a variable transformation $y = nx^n$ that rendered the integral as the sum of two integrals over $[0,1]$ and $[1,n]$, each of which tended to 1. Lau made a transformation $y = n^{1/n}x$. Two incorrect solutions were received.

3699. [2011 : 542, 544] *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC denote a triangle, I its incenter, and ρ_a, ρ_b, and ρ_c the inradii of IBC, ICA, and IAB, respectively. Prove that

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} \ \geq \ \frac{18\tan(75^\circ)}{a+b+c}\,.$$

I. Solution by George Apostolopoulos, Messolonghi, Greece.

Let $2\alpha = B+C$, $2\beta = C+A$ and $2\gamma = A+B$. Then $\alpha+\beta+\gamma = 180^\circ$, $a = r(\tan\beta + \tan\gamma)$, $b = r(\tan\gamma + \tan\alpha)$, $c = r(\tan\alpha + \tan\beta)$, $IB = r\sec\beta$, $IC = r\sec\gamma$ and $2[IBC] = ar$ and $2[IBC] = ar$, so that

$$\frac{1}{\rho_a} = \frac{1}{r} + \frac{\cos\beta + \cos\gamma}{r\sin\alpha}$$

with analogous expressions for $1/\rho_b$ and $1/\rho_c$.

Observe that $\tan 75^\circ = \sqrt{3}+2$ and $\tan\alpha\cdot\tan\beta\cdot\tan\gamma = \tan\alpha+\tan\beta+\tan\gamma \geq 3\tan 60^\circ = 3\sqrt{3}$ (from the convexity of the tangent function). Then

$$\frac{18\tan 75^\circ}{a+b+c} = \frac{9\sqrt{3}+18}{r\tan\alpha\cdot\tan\beta\cdot\tan\gamma}.$$

Thus, it is required to show that

$$3\tan\alpha\cdot\tan\beta\cdot\tan\gamma + \sum_{\text{cyclic}}\left(\frac{\cos\beta+\cos\gamma}{\cos\alpha}\right)\tan\beta\cdot\tan\gamma \geq 9\sqrt{3}+18.$$

This inequality holds for the two terms respectively, that for the second term relying on an application of the Arithmetic-Geometric Means inequality.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

We note that $\tan 75^\circ = \tan(45^\circ + 30^\circ) = 2+\sqrt{3}$. Let r, R and s denote the inradius, circumradius and semiperimeter, respectively, of triangle ABC. Let h_I, h_A, h_B and $[IAB]$ denote the altitudes and area of triangle IAB. We have that $h_I = r$, $h_A = c\sin(B/2)$ and $h_B = c\sin(A/2)$. Hence

$$\frac{1}{\rho_a} = \frac{AB+AI+BI}{2[IAB]} = \frac{1}{h_I} + \frac{1}{h_B} + \frac{1}{h_A} = \frac{1}{r} + \frac{1}{c\sin(A/2)} + \frac{1}{c\sin(B/2)}.$$

Similar identities hold for $1/\rho_b$ and $1/\rho_c$. We find that

$$\begin{aligned}&(a+b+c)\left(\frac{1}{\rho_a}+\frac{1}{\rho_b}+\frac{1}{\rho_c}\right)\\ &\qquad=\frac{3(a+b+c)}{r}+(a+b+c)\left(\frac{1}{a\sin(B/2)}+\frac{1}{b\sin(C/2)}+\frac{1}{c\sin(A/2)}\right)\\ &\qquad+(a+b+c)\left(\frac{1}{a\sin(C/2)}+\frac{1}{b\sin(A/2)}+\frac{1}{c\sin(B/2)}\right).\end{aligned}$$

By the Arithmetic-Geometric Means inequality, we have that

$$\begin{aligned}\frac{3(a+b+c)}{r}&=\frac{6s}{\sqrt{(s-a)(s-b)(s-c)/s}}\\ &\geq\frac{6s^{3/2}}{(s/3)^{3/2}}=6\cdot 3^{3/2}=18\sqrt{3}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, the Arithmetic-Geometric Means inequality and Euler's inequality $R\geq 2r$ in succession, we obtain that

$$\begin{aligned}&(a+b+c)\left(\frac{1}{a\sin(B/2)}+\frac{1}{b\sin(C/2)}+\frac{1}{c\sin(A/2)}\right)\\ &\qquad\geq\left(\frac{1}{\sqrt{\sin(A/2)}}+\frac{1}{\sqrt{\sin(B/2)}}+\frac{1}{\sqrt{\sin(C/2)}}\right)^2\\ &\qquad\geq\frac{9}{\sqrt[3]{\sin(A/2)\sin(B/2)\sin(C/2)}}=9\sqrt[3]{\frac{4R}{r}}\geq 18.\end{aligned}$$

Analogously,

$$(a+b+c)\left(\frac{1}{a\sin(C/2)}+\frac{1}{b\sin(A/2)}+\frac{1}{c\sin(B/2)}\right)\geq 18.$$

Consequently,

$$(a+b+c)\left(\frac{1}{\rho_a}+\frac{1}{\rho_b}+\frac{1}{\rho_c}\right)\geq 18(2+\sqrt{3})=18\tan 75^\circ.$$

Equality holds if and only if triangle ABC is equilateral.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania and TITU ZVONARU, Comăneşti, Romania (joint); and the proposer.

3700. [2011 : 542, 544] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle and $a = BC$, $b = CA$, $c = AB$. Given that

$$aPA^2 + cPB^2 + bPC^2 = cPA^2 + bPB^2 + aPC^2 = bPA^2 + aPB^2 + cPC^2$$

for some point P, show that $\triangle ABC$ is equilateral.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The problem is missing a necessary restriction; the correct statement (extended to three-dimensional space by the editors): *Let ℓ be the line in Euclidean space that is that is perpendicular to the plane of the given triangle ABC and passes through its circumcenter; if there exists a point not on ℓ that satisfies the given equations, then the triangle must be equilateral.*

Set $x = PA, y = PB$, and $z = PC$, and assume the labels have been chosen so that $z \neq 0$. The given system of equations then becomes

$$a(x^2-z^2)+b(z^2-y^2)+c(y^2-x^2) = 0 \text{ and } a(z^2-y^2)+b(y^2-x^2)+c(x^2-z^2) = 0,$$

or, equivalently,

$$ax^2 - by^2 + cy^2 - cx^2 = (a-b)z^2 \tag{1}$$

$$-ay^2 + by^2 - bx^2 + cx^2 = (c-a)z^2 \tag{2}$$

Case 1. If $a \neq b, c$, then we can multiply the first equation by $c-a$, the second by $a-b$, and subtract to obtain

$$(c-a)(ax^2 - by^2 + cy^2 - cx^2) - (a-b)(-ay^2 + by^2 - bx^2 + cx^2) = 0,$$

or, after regrouping,

$$(x^2 - y^2)(bc + ca + ab - a^2 - b^2 - c^2) = 0.$$

Because $a \neq b, c$ the AM-GM inequality implies that $bc+ca+ab-a^2-b^2-c^2 \neq 0$. Hence, $x^2 = y^2$, and (from (1))

$$z^2 = \frac{ax^2 - by^2 + cy^2 - cx^2}{a-b} = \frac{x^2(a-b)}{a-b} = x^2.$$

Consequently, $PA = PB = PC$; that is, P is equidistant from the three vertices so that it lies on the line ℓ. Observe that when $PA = PB = PC$, each of the three expressions in the problem statement are equal to $PA^2(a+b+c)$.

Case 2. If $a = b$ then equations (1) and (2) become

$$(a-c)(x^2-y^2) = 0 \text{ and } (c-a)(x^2-z^2) = 0.$$

If in addition, $x = y = z$, then again, P lies on ℓ. Otherwise, $c = a = b$ and ΔABC is equilateral, as claimed.

Also solved by MARIAN DINCĂ, Bucharest, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; M. A. PRASAD, India; and the proposer.

The solution submitted by the proposer clearly indicates that he had intended to exclude the circumcentre as a possible position for P (as in the restatement of the problem). Among the other three submissions, two dealt with specialized interpretations, and one simply provided the counterexample.