RECURRING CRUX CONFIGURATIONS 9

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Triangles Whose Angles Satisfy $B = 90° + C$

The main property of the triangles we look at this month first appeared in **Crux** in 2000:

Problem 2525 (April) [2000: 177; 2001: 270-271] (proposed by Antreas P. Hatzipolakis and Paul Yiu; reworded here). In a triangle ABC with $\angle B > \angle C$,

> $\angle B = 90° + \angle C$ *if and only if the centre of the nine-point circle lies on the line* BC.

The proposers also called for the reader to show that if, in addition, $\angle A = 60°$, then the 9-point centre N is the point where the bisector of angle A intersects BC. Of course, as we saw in Part 3 of this column, AN bisects angle A whenever $\angle A = 60°$, whatever B and C might be.

Problem 2765 [2002: 397; 2003: 349-351] (Proposed by K.R.S. Sastry). Derive a set of side-length expressions for the family of Heron triangles ABC in which the nine-point centre N lies on the line BC.

Michel Bataille's solution invoked Problem 2525; specifically, he proved that ΔABC has integer sides and area while $\angle B = 90° + \angle C$ if and only if there exists a primitive Pythagorean triple (m, n, k) with $m > n$ and a positive integer d for which

$$a = d(m^2 - n^2), \quad b = dkm, \quad c = dkn.$$

The area, given by Heron's fromula, equals $d^2mn(m^2 - n^2)/2$, which is an integer since m and n have opposite parity. The original problem required further that N lie in the interior of the segment BC; this will be the case if $m^2 > 3n^2$. For an explicit example having B between N and C we can set $m = 4, n = 3, k = 5$, and $d = 1$; the resulting triangle has $a = 7, b = 20, c = 15, \sin B = 4/5, \sin C = 3/5$, and area $= 42$.

As part of his solution, Sastry added a list of nine properties that are equivalent for any triangle ABC:

(1) The nine-point centre is on BC.

(2) $|B - C| = 90°$.

(3) $\tan B \tan C = -1$.

(4) $OA||BC$ (where O is the circumcentre of ΔABC).

(5) AH is tangent to the circumcircle of ΔABC (where H is the orthocentre).

(6) AH is tangent to the nine-point circle of ΔABC.

(7) BC bisects segment AH.

(8) $AN = \frac{1}{2}OH$.

(9) AC and BC trisect $\angle OCH$.

Problem 2867 [2003: 399; 2004: 378-379] (proposed by Antreas P. Hatzipolakis and Paul Yiu). With vertices B and C of triangle ABC fixed and its nine-point centre sliding along the line BC, the locus of the vertex A is the rectangular hyperbola (excluding B and C) whose major axis is BC.

When $B = C + 90°$ every point except B on the branch of the hyperbola through B is the vertex of such a triangle; when $C = B + 90°$, $A \neq C$ sweeps out the branch through C. Christopher J. Bradley remarked that this is the hyperbolic analogue of the familiar theorem about angles inscribed in semicircles: When BC is the diameter of a semicircle containing A, then $B + C = 90°$; when BC is the major axis of a rectangular hyperbola containing A, then $|B - C| = 90°$.

I conclude this month's essay, as well as this series, by recalling an observation made by Charles W. Trigg [1978: 79] that was included in Part 2 of the series: If the sides of ΔABC satisfy $c + a = 2b$ while the angles satisfy $A = C + 90°$, then the sides a, b, c are in the ratio $(\sqrt{7} + 1) : \sqrt{7} : \sqrt{7} - 1$.

Call for Problem of the Month Articles

We have introduced a few new features this volume. One of these new features is the *Problem of the Month* which is dedicated to the memory of former ***CRUX with MAYHEM*** Editor-in-Chief Jim Totten. The *Problem of the Month* features a problem and solution that we know Jim would have liked. Do you have a favourite problem that you would like to share? Write it up and send it to the editor at crux-editors@cms.math.ca. Articles featured in the *Problem of the Month* should be instructive, anecdotal and entertaining.

We are also seeking an editor for this column. If you are interested, or have someone to recommend, please contact the editor at the email address above.