THE OLYMPIAD CORNER

No. 308

Nicolae Strungaru

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 avril 2014**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC106. Déterminer tous les entiers positifs n pour lesquels tous les entiers à n positions décimales, contenant $n-1$ uns et 1 seul sept, sont premiers.

OC107. Le périmètre du triangle ABC est égal à 4. Des points X et Y sont placés sur les arcs AB et AC de façon à ce que $AX = AY = 1$. Les segments BC et XY s'entrecoupent au point M, à leur intérieur. Démontrer que le périmètre de l'un des triangles ABM et AMC est égal à 2.

OC108. Déterminer toutes les fonctions $f : \mathbb{R} \mapsto \mathbb{R}$ telles que

$$2f(x) = f(x+y) + f(x+2y)$$

pour tout $x \in \mathbb{R}$, $y \in [0, \infty)$.

OC109. Soit $a_1, a_2, \ldots, a_n, \ldots$ une permutation des entiers positifs. Démontrer qu'il existe infiniment d'entiers positifs i tels que $\text{pgcd}(a_i, a_{i+1}) \leq \frac{3}{4}i$.

OC110. Soit G un graphe qui ne contient pas K_4 comme sous graphe. Si le nombre de sommets de G est $3k$, où k est entier, quel est le nombre maximal de triangles dans G ?

. . . .

OC106. Find all the positive integers n for which all the n digit integers containing $n-1$ ones and 1 seven are prime.

OC107. The perimeter of triangle ABC is equal to 4. Points X and Y are marked on the rays AB and AC in such a way that $AX = AY = 1$. The segments BC and XY intersect at point M in their interior. Prove that the perimeter of one of the triangles ABM or AMC is equal to 2.

OC108. Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$2f(x) = f(x+y) + f(x+2y)$$

for all $x \in \mathbb{R}$, $y \in [0, \infty)$.

OC109. Let $a_1, a_2, \ldots, a_n, \ldots$ be a permutation of the set of positive integers. Prove that there exist infinitely many positive integers i such that $\gcd(a_i, a_{i+1}) \leq \frac{3}{4}i$.

OC110. Let G be a graph, not containing K_4 as a subgraph. If the number of vertices of G is $3k$, for some integer k, what is the maximum number of triangles in G?

OLYMPIAD SOLUTIONS

OC46. Let p be a prime number, and let x, y, z be integers so that $0 < x < y < z < p$. Suppose that x^3, y^3 and z^3 have the same remainders when divided by p. Prove that $x^2 + y^2 + z^2$ is divisible by $x + y + z$.
(*Originally question 5 from the 2009 Singapore Mathematical Olympiad, open section, round 2.*)

Solved by Michel Bataille, Rouen, France ; Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Oliver Geupel, Brühl, NRW, Germany ; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the similar solutions of Bataille and Manes.

Let $x^3 \equiv y^3 \equiv z^3 \equiv a \pmod{p}$. Then the polynomial $P(w) = w^3 - a$ factors as

$$w^3 - a \equiv (w-x)(w-y)(w-z) \equiv w^3 - (x+y+z)w^2 + (xy+xz+yz)w - xyz \pmod{p} .$$

Thus

$$x + y + z \equiv xy + xz + yz \equiv 0 \pmod{p} .$$

Also

$$x^2 + y^2 + z^2 \equiv (x+y+z)^2 - 2(xy + xz + yz) \equiv 0 \pmod{p} .$$

As $0 < x+y+z < 3p$ we have $x+y+z = p$ or $x+y+z = 2p$. Then if $x+y+z = p$ the claim is obvious, while if $x + y + z = 2p$ it follows that $x^2 + y^2 + z^2$ is also even, and hence divisible by $2p = x + y + z$.

OC47. Let a, b be two distinct odd positive integers. Let a_n be the sequence defined as $a_1 = a$; $a_2 = b$; $a_n =$ the largest odd divisor of $a_{n-1} + a_{n-2}$. Prove that there exists a natural number N so that, for all $n \geq N$ we have $a_n = \gcd(a, b)$.
(*Originally question 7 from the 2009 India IMO selection test.*)

Solved by Arkady Alt, San Jose, CA, USA and Titu Zvonaru, Cománeşti, Romania. We give the solution of Zvonaru.

Let $d = \gcd(a, b)$. Since $a_{n-1} + a_{n-2} = 2^{\beta} a_n$, for some $\beta \geq 1$, we can easily deduce that $d \mid a_n$ for all n by induction. By induction it is also easy to show that $\gcd(a_n, a_{n-1}) = \gcd(a, b) = d$. Let $m = \max\{a, b\}$. Then for each n, as $a_{n-1} + a_{n-2}$ is even we have

$$a_n \leq \frac{a_{n-1} + a_{n-2}}{2} .$$

It follows immediately by induction that $a_n \leq m$ for all n.

Claim: If $a_k < a_{k+1}$ for some k then

$$a_n < a_{k+1} \text{ for all } n > k + 1 .$$

We prove this by induction. Suppose that $a_k < a_{k+1}$ for some k then

$$a_{k+2} \leq \frac{a_{k+1} + a_k}{2} < \frac{a_{k+1} + a_{k+1}}{2} = a_{k+1}.$$

Similarly,

$$a_{k+3} \leq \frac{a_{k+2} + a_{k+1}}{2} < \frac{a_{k+1} + a_{k+1}}{2} = a_{k+1},$$

so the claim is true for $n = k + 2$ and $n = k + 3$. Suppose the claim is true for $n = j$ and $n = j + 1$ for some $j \geq k + 2$, then

$$a_{j+2} \leq \frac{a_{j+1} + a_j}{2} < \frac{a_{k+1} + a_{k+1}}{2} = a_{k+1} .$$

Thus the claim is true for all $n > k + 1$.

It follows from here that there can only be finitely many k for which $a_k < a_{k+1}$. Indeed, assume by contradiction that we can find infinitely many $i_1 < i_2 < \cdots < i_n < \cdots$ so that

$$a_{i_j} < a_{i_j+1} .$$

Then, as $i_{j+1} + 1 > i_j + 1$ it follows from the claim that $a_{i_{j+1}+1} < a_{i_j+1}$ and hence

$$a_{i_1+1} > a_{i_2+1} > \cdots > a_{i_j+1} > \cdots$$

is a strictly decreasing infinite sequence of positive integers, contradiction.

Thus, there are only finitely many k for which $a_k < a_{k+1}$. Hence, there exists a largest such k. Thus, if k_0 denotes the largest such k, we have

$$a_n \geq a_{n+1} \text{ for all } n \geq k_0 + 1 .$$

As a_n is decreasing from $n = k$, and as it is positive, it cannot be strictly decreasing. Thus, there exists an $m \geq k$ so that $a_m = a_{m+1}$.

As $\gcd(a_m, a_{m+1}) = d$ we get that $a_m = a_{m+1} = d$, and then it is easy to prove by induction that $a_n = d$ for all $n \geq m$.

OC48. The angles of a triangle ABC are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. The bisectors meet the opposite sides at A', B' and C'. Prove the $A'B'C'$ is an isosceles triangle.
(*Originally question 2 from the 2009 Columbia Mathematical Olympiad.*)

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA and Titu Zvonaru, Comănești, Romania. We give the solution of Bataille.

We suppose $\angle BAC = \frac{4\pi}{7}, \angle ABC = \frac{2\pi}{7}, \angle BCA = \frac{\pi}{7}$ and we denote by a, b, c the side lengths BC, CA, AB respectively. We show that $A'B' = A'C'$.

The law of sines yields

$$\frac{a}{\sin\left(\frac{4\pi}{7}\right)} = \frac{b}{\sin\left(\frac{2\pi}{7}\right)} = \frac{c}{\sin\left(\frac{\pi}{7}\right)}.$$

Thus

$$\cos\left(\frac{2\pi}{7}\right) = \frac{\sin\left(\frac{4\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = \frac{a}{2b}.$$

Similarly

$$\cos\left(\frac{\pi}{7}\right) = \frac{b}{2c}.$$

From the bisector theorem we have

$$\frac{BC'}{a} = \frac{C'A}{b} = \frac{c}{a+b}; \quad \frac{BA'}{c} = \frac{CA'}{b} = \frac{a}{b+c}; \quad \frac{AB'}{c} = \frac{B'C}{a} = \frac{b}{c+a},$$

thus

$$BC' = \frac{ca}{a+b}; \quad BA' = \frac{ca}{b+c}; \quad CA' = \frac{ab}{b+c}; \quad CB' = \frac{ab}{c+a}.$$

Let I be the incentre of ABC. Then

$$\frac{AI}{AA'} = \frac{b+c}{a+b+c}; \qquad \frac{BI}{BB'} = \frac{a+c}{a+b+c}.$$

As $\angle ABA' = \frac{2\pi}{7} = \angle AB'B$ we get $IA = IB'$. Similarly $AA' = A'B$ and $BB' = B'C$. Thus

$$BI = \frac{a+c}{a+b+c}B'C = \frac{ab}{a+b+c}$$
$$IB' = IA = \frac{b+c}{a+b+c}AA' = \frac{c\,a}{a+b+c}$$

and adding these we get

$$\frac{ab}{a+c} = B'C = BB' = BI + IB' = \frac{a(b+c)}{a+b+c}.$$

Hence

$$b(a+b+c) = (a+c)(b+c) \Rightarrow b^2 = ac + c^2.$$

Also, $\cos(\frac{2\pi}{7}) = 2\cos^2(\frac{\pi}{7}) - 1$ yields

$$\frac{a}{2b} = \frac{b^2}{2c^2} - 1 \Rightarrow b^3 = ac^2 + 2bc^2 .$$

Combining the last two relations we get

$$ab = ac + bc .$$

Then, we get

$$BC' = \frac{c^2}{b}; \quad BA' = \frac{bc}{a}; \quad CA' = \frac{b^2}{a}; \quad CB' = \frac{ac}{b}$$

and hence

$$\begin{aligned} A'C'^2 &= BA'^2 + BC'^2 - 2BA' \cdot BC' \cos\frac{2\pi}{7} = \frac{b^2c^2}{a^2} + \frac{c^4}{b^2} - \frac{c^3}{b} , \\ A'B'^2 &= CA'^2 + CB'^2 - 2CA' \cdot CB' \cos\frac{\pi}{7} = \frac{b^4}{a^2} - b^2 + \left(\frac{b^2 - c^2}{b}\right)^2 \\ &= \frac{b^4}{a^2} + \frac{c^4}{b^2} - 2c^2 . \end{aligned}$$

Then,

$$\begin{aligned} A'C'^2 - A'B'^2 &= \frac{c}{ab}(b^3 - 2abc + ac^2) = \frac{c}{ab}(b^3 - 2(b^3 - bc^2) + ac^2) \\ &= \frac{c}{ab}(2bc^2 - b^3 + ac^2) = 0 . \end{aligned}$$

Thus $A'B' = A'C'$ and we are done.
[*Ed.*: Covas mentioned that a proof can be found in Leon Bankoff and Jack Garfunkel, *The Heptagonal Triangle*, Mathematics Magazine 46 (1973), p. 17.]

OC49. Let N be a positive integer. How many non-congruent triangles are there, whose vertices lie on the vertices of a regular $6N$-gon?
(*Originally question 11 from the 2009 India IMO selection test.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We claim there are $3N^2$ non-congruent triangles.

Draw the regular polygon on a circle of radius R so that the arc length between two consecutive vertices is 1. Then the circle has length $6N$. It is easy to see that two triangles with the vertices on the $6N$ points are congruent if and only if they have the same corresponding arc lengths on the circle. Thus, the number of non-congruent triangles is equal to the number of unordered partitions of $6N$ into 3 (not necessarily distinct) summands.

Thus, denoting by x, y, z the arc lengths, without loss of generality we have

$x \le y \le z$ and $x + y + z = 6N$. Thus, the number $g(N)$ of triangles becomes

$$\begin{aligned} g(N) &= \#\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x \le y\,,\ x + y \le 6N - y\} \\ &= \#\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x \le y\,,\ x + 2y \le 6N\} \\ &= \sum_{x=1}^{2N} \left(\left\lfloor \frac{6N - x}{2} \right\rfloor - (x-1) \right) \\ &= \sum_{\substack{x\,=\,1 \\ x \text{ is odd}}}^{2N-1} \left(3N - \frac{x+1}{2} - (x-1) \right) + \sum_{\substack{x\,=\,1 \\ x \text{ is even}}}^{2N} \left(3N - \frac{x}{2} - (x-1) \right) \\ &= \sum_{k=1}^{N} (3N - k - (2k-2)) + \sum_{k=1}^{N} (3N - k - (2k-1)) \\ &= \sum_{k=1}^{N} (6N - 6k + 3) \\ &= 6N^2 - 6\frac{N(N+1)}{2} + 3N = 6N^2 - 3N^2 - 3N + 3N = 3N^2 \end{aligned}$$

[*Ed.*: The stars and stripes method tell us that there are $\binom{6N-1}{3-1} = \frac{(6N-1)(6N-2)}{2}$ triples of (x, y, z) satisfying the equation. From those, 1 has $x = y = z$ and $3N - 2$ satisfy $x = y \neq z$ (the cases $x = y = 3N, z = 0$ and $x = y = z = 2N$ need to be eliminated). Thus by permutations, $9N - 6$ have exactly two values equal. The remaining $\frac{(6N-1)(6N-2)}{2} - 9N + 6 - 1$ have three distinct values. Counting now the unordered triples, the one with $(x = y = z)$ is one, the $9N - 6$ come in groups of three up to permutations, and the remaining $\frac{(6N-1)(6N-2)}{2} - 9N + 6 - 1$ come in groups of 6. Thus we have

$$\begin{aligned} &\frac{\frac{(6N-1)(6N-2)}{2} - 9N + 6 - 1}{6} + \frac{9N-6}{3} + 1 \\ &\qquad = \frac{36N^2 - 18N + 2 - 18N + 10}{12} + 3N - 2 + 1 \\ &\qquad = 3N^2 - 3N + 1 + 3N - 1 = 3N^2\,. \end{aligned}$$]

OC50. Let $n \ge 2$. If n divides $3^n + 4^n$, prove that 7 divides n.
(*Originally question 8 from the 2009 India IMO selection test.*)

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

As $3^n + 4^n$ is odd, it follows that n must be odd. Moreover, as $3^n + 4^n \equiv 1 \pmod 3$ it follows that $3 \nmid n$. Thus n is an odd integer greater than 3. Let p be the smallest prime dividing n.

Let

$$a \equiv 4 \cdot 3^{-1} \pmod p\,.$$

Then

$$0 \equiv 3^n + 4^n \equiv 3^n(1 + a^n) \pmod p .$$

As 3 is invertible mod p, we get

$$a^n \equiv -1 \pmod p .$$

As n is odd, this implies

$$(-a)^n \equiv 1 \pmod p .$$

Let r be the order of $-a$ modulo p. Then $r \mid n$ and $r \mid p-1$. As p is the smallest divisor of n, $r < p$ and $r \mid n$, it follows that $r = 1$. Hence

$$1 \equiv -a \equiv -4 \cdot 3^{-1} \pmod p$$

and hence

$$3 \equiv -4 \pmod p .$$

Thus $p \mid 7$ and hence $p = 7$.

Unsolved Crux Problem

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Recently, Chris Fisher published a list of unsolved problems from *Crux* [2010 : 545, 547]. Below is one of these unsolved problems. Note that the solution to part (a) has been published [2005 : 468-470] but (b) remains open.

2977. [2004 : 429, 432; 2005 : 468-470] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a_1, a_2, $\ldots$, a_n be positive real numbers, let $r = \sqrt[n]{a_1 a_2 \cdots a_n}$, and let

$$E_n = \frac{1}{a_1(1+a_2)} + \frac{1}{a_2(1+a_3)} + \cdots + \frac{1}{a_n(1+a_1)} - \frac{n}{r(1+r)} .$$

(a) Prove that $E_n \geq 0$ for

- (a_1) $n = 3$;
- (a_2) $n = 4$ and $r \leq 1$;
- (a_3) $n = 5$ and $\frac{1}{2} \leq r \leq 2$;
- (a_4) $n = 6$ and $r = 1$.

(b)★ Prove or disprove that $E_n \geq 0$ for

- (b_1) $n = 5$ and $r > 0$;
- (b_2) $n = 6$ and $r \leq 1$.