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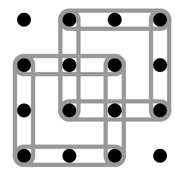
Téléc : 613–733–8994
email : abonnements@smc.math.ca

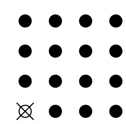
MAYHEM SOLUTIONS

M507. *Proposed by the Mayhem Staff.*

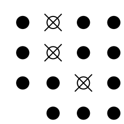


A 4 by 4 square grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form squares, two different 2 by 2 squares are shown in the diagram. What is the least number of pegs that must be removed so that **no** squares can be formed?

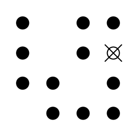
Solution by Juz'an Nari Haifa, student, SMPN 8, Yogyakarta, Indonesia.



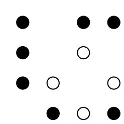
To find the least number of pegs that must be removed, we will take pegs away as necessary. We must consider squares of side length 3, 2, 1, $\sqrt{2}$ and $\sqrt{3}$. We will begin with a 3 by 3 square by eliminating the bottom left peg. Consequently, no 3 by 3 square can be formed.



Next we will remove any 2 by 2 squares. There are four different 2 by 2 squares, but one has already been eliminated. Since none of them share vertices, we need to remove 3 squares. Since we are looking for the smallest number of pegs, we will not remove any from the corners so we are left with the pegs shown in the diagram to the right.



At this point there is only one 1 by 1 square left so we will proceed by removing the bottom right peg of that square. Due to the pegs we chose to remove so far, neither of the two $\sqrt{5}$ by $\sqrt{5}$ squares can be formed.



This leaves us with only the $\sqrt{2}$ by $\sqrt{2}$ squares to consider. There is only one $\sqrt{2}$ by $\sqrt{2}$ left (marked with the open circles in the diagram) and we can remove any one of its pegs to remove the square. Thus we have removed 6 pegs and no more squares can be formed.

Note that we can apply the same method to obtain different answers, but the minimum number of pegs that need to be removed will always be 6.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; MUHAMMAD LABIB IRFANUDDIN, student, SMP N 8 YOGYAKARTA, Indonesia; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; TAUPIEK DIDA PALLEVI, student, SMP N 8 YOGYAKARTA, Indonesia; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and ARIF SETYAWAN, student, SMP N 8 YOGYAKARTA, Indonesia. One incorrect solution was submitted.

M508. *Proposed by the Mayhem Staff.*

In 1770, Joseph Louis Lagrange proved that **every** non-negative integer can be expressed as the sum of the squares of four integers. For example $6 = 2^2+1^2+1^2+0^2$ and $27 = 5^2+1^2+1^2+0^2 = 4^2+3^2+1^2+1^2 = 3^2+3^2+3^2+0^2$ (in the theorem it is acceptable to use 0^2, or to use a square more than once). Notice that 27 had several different representations. Show that there is a number, not greater than 1 000 000 that can be represented as a sum of the squares of four **distinct non-negative** integers in more than 100 ways. (Note that rearrangements are not considered different, so $4^2+3^2+2^2+1^2 = 1^2+2^2+3^2+4^2$ are the same representation of 30.)

Solution by Florencio Cano Vargas, Inca, Spain.

We are going to count the number of combinations of squares of four integers, $(x_i^2, i = 1, 2, 3, 4)$, whose sum does not exceed 1 000 000. To get a conservative bound let us consider the case where all four integers are equal, i.e. $x_1 = x_2 = x_3 = x_4 \equiv x$. Then we have, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4x^2 \le 10^6 \Rightarrow x \le 500$. For any combination four non-negative integers not greater than 500, the sum of the squares of these four numbers is less than 1 000 000. Note that there are some combinations, like $1^2 + 2^2 + 3^2 + 600^2 = 360\ 014 < 1\ 000\ 000$, with some $x_i > 500$ but they are not included in our counting. Therefore we will limit ourselves to combinations $x_1^2 + x_2^2 + x_3^2 + x_4^2$ with $0 \le x_i \le 500$ and $x_i \ne x_j$ when $i \ne j$. Since each x_i can take 501 values, the number of combinations is given by $\binom{501}{4}$. [**Note**: If we had allowed combinations with repeated integers, we would have combinations with repetition, i.e. we would have had $\binom{504}{4}$ combinations to deal with].

We then have $\binom{501}{4} \approx 2.5937 \cdot 10^9$ combinations to assign to at most 1 000 000 numbers. This gives an average of

$$\binom{501}{4} \div 10^6 \approx 2593.7$$

combinations per integer, hence there is at least one integer not greater than 1 000 000 that can be represented as a sum of four distinct non-negative integer in 2594 ways by the pigeonhole principle.

Note: Following this method we can easily prove that we can find a number not greater than N that can be represented as a sum of squares of four integers in more than 100 ways, where N is the minimal solution of the inequality

$$\binom{\left\lfloor\sqrt{\frac{N}{4}}\right\rfloor + 1}{4} \div N \ge 100$$

which yields $N = 40\ 000$. It should be noted that one could find a better bound with a more refined counting method since we have discarded integers x_i which are larger than $\left\lfloor \frac{\sqrt{N}}{2} \right\rfloor$.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA. One incorrect solution was submitted.

This problem was inspired by an example on page 22 of the book Numbers: A Very Short Introduction *by Peter M. Higgins, Oxford University Press, 2011. In the example the author showed that there is a number less than* 4 000 000 000 *that can be written as the sum of four different* cubes *in 10 distinct ways.*

M509. *Proposed by Titu Zvonaru, Comăneşti, Romania.*

Let ABC be a triangle with angles B and C acute. Let D be the foot of the altitude from vertex A. Let E be the point on AC such that $DE \perp AC$ and let M be the midpoint of DE. Show that if $AM \perp BE$, then $\triangle ABC$ is isosceles.

Solution by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania.

We will begin by setting up a set of coordinates such that the side BC is on the x-axis and the altitude AD is on the y-axis. Then we have $A(0,a)$, $B(-b,0)$, $C(c,0)$ and $D(0,0)$, where $a,b,c \in (0,\infty)$. The equation of line AC is $y = -\frac{a}{c}x + a$ and since $DE \perp AC$, the equation of the line DE is $y = \frac{c}{a}x$. Their point of intersection is $E\left(\frac{a^2c}{a^2+c^2}, \frac{ac^2}{a^2+c^2}\right)$.

The midpoint of DE is $M\left(\frac{a^2c}{2(a^2+c^2)}, \frac{ac^2}{2(a^2+c^2)}\right)$. The slopes of the lines AM and BE are respectively

$$m_{AM} = \frac{y_M - y_A}{x_M - x_A} = -\frac{2a^2+c^2}{ac}, \quad m_{BE} = \frac{y_E - y_B}{x_E - x_B} = \frac{ac^2}{a^2c + b(a^2+c^2)}.$$

These lines are perpendicular if and only if their slopes satisfy $m_{AM} \cdot m_{BE} = -1$. Thus

$$c(a^2+c^2) = b(a^2+c^2),$$

and since $a,b,c > 0$, we get $b = c$, hence it follows that triangle ABC is isosceles.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ADRIAN NACO, Polytechnic University of Tirana, Albania; RICARD PEIRÓ I ESTRUCH, IES "Abastos", Valencia, Spain; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; MIHAÏ STOËNESCU, Bischwiller, France; KONSTANTINE ZELATOR, University of Pittsburgh, Pittsburgh, PA, USA; and the proposer. One incorrect solution was submitted.

M510. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

If $a,b,c \in \mathbb{C}$ such that $|a| = |b| = |c| = r > 0$ and $a+b+c \neq 0$, compute the value of the expression

$$\frac{|ab+bc+ca|}{|a+b+c|}$$

in terms of r.

Solution by Adrian Naco, Polytechnic University of Tirana, Albania.

Based on the properties of the complex numbers, we have that

$$\begin{aligned} 0 \neq |a+b+c|^2 &= (a+b+c)\overline{(a+b+c)} \\ &= a\overline{a} + b\overline{b} + c\overline{c} + (a\overline{b} + \overline{a}b) + (b\overline{c} + \overline{b}c) + (c\overline{a} + \overline{c}a) \\ &= |a|^2 + |b|^2 + |c|^2 + (a\overline{b} + \overline{a}b) + (b\overline{c} + \overline{b}c) + (c\overline{a} + \overline{c}a) \\ &= 3r^2 + (a\overline{b} + \overline{a}b) + (b\overline{c} + \overline{b}c) + (c\overline{a} + \overline{c}a)\,. \qquad (1) \end{aligned}$$

Furthermore,

$$\begin{aligned}|ab+bc+ca|^2 &= (ab+bc+ca)\overline{(ab+bc+ca)}\\ &= ab\overline{ab}+bc\overline{bc}+ca\overline{ca}\\ &\qquad +(ab\overline{bc}+\overline{ab}bc)+(bc\overline{ca}+\overline{bc}ca)+(ca\overline{ab}+\overline{ca}ab)\\ &= |a|^2|b|^2+|b|^2|c|^2+|c|^2|a|^2\\ &\qquad +|b|^2(a\overline{c}+\overline{a}c)+|c|^2(b\overline{a}+\overline{b}a)+|a|^2(c\overline{b}+\overline{c}b)\\ &= 3r^4+r^2(c\overline{a}+\overline{c}a)+r^2(b\overline{a}+\overline{b}a)+r^2(c\overline{b}+\overline{c}b)\\ &= r^2[3r^2+(c\overline{a}+\overline{c}a)+(b\overline{a}+\overline{b}a)+(c\overline{b}+\overline{c}b)]\,. \qquad (2)\end{aligned}$$

From (1) and (2) we get

$$\begin{aligned}|ab+bc+ca|^2 = r^2|a+b+c|^2 &\Rightarrow |ab+bc+ca| = r|a+b+c|\\ &\Rightarrow \frac{|ab+bc+ca|}{|a+b+c|} = r\,.\end{aligned}$$

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; IOAN VIOREL CODREANU, Secondary School student, Satulung, Maramureş, Romania; MARIAN DINCĂ, Bucharest, Romania; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ANDHIKA GILANG, student, SMPN 8, Yogyakarta, Indonesia; THARIQ SURYA GUMELAR, student, SMPN 8, Yogyakarta, Indonesia; CORNELIU MĂNESCU-AVRAM, Transportation High School, Ploieşti, Romania; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRÓ I ESTRUCH, IES "Abastos", Valencia, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; HENRY RICARDO, Tappan, NY, USA; BRUNO SALGUEIRO FANEGO, Viveiro, Spain; and the proposer.

M511. *Proposed by Gili Rusak, student, Shaker High School, Latham, NY, USA.*

Pens come in boxes of 48 and 61. What is the smallest number of pens that can be bought in two ways if you must buy at least one box of each type?

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

Suppose we buy a boxes with 48 pens each and b boxes with 61 pens each, and in another way we buy c boxes with 48 pens each and d boxes with 61 pens each. Then we have $48a+61b$ in one way and $48c+61d$ in the other way. Since we want the number of pens to be the same in both ways, we have $48a+61b = 48c+61d$. If $a = c$ then $b = d$, in which case the two ways are the same. Since we set up two different ways of buying the pens, then we have $a \neq c$.

Since 48 and 61 are relatively prime, then from $48(a-c) = 61(b-d)$ it follows that 61 divides $48(a-c)$. Without loss of generality, we will suppose that $a > c$. It follows that the minimum possible value of $a-c$ is 61, and hence $48(c+61)+61b = 48c+61d$, or equivalently, $d = 48+b$.

Since we must buy at least one box of each type, the smallest possible value of b is $b = 1$ and then $d = 49$. Similarly, since the smallest possible value of c is

$c = 1$, it follows that $a = 62$. Therefore, the smallest number of pens that can be bought is $(48)(62) + (61)(1) = (48)(1) + (61)(49) = 3037$.

Also solved by FLORENCIO CANO VARGAS, Inca, Spain; GLORIA (YULIANG) FANG, University of Toronto Schools, Toronto, ON; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; MISSOURI STATE UNIVERSITY PROBLEM SOLVING GROUP, Springfield, MO, USA; RICARD PEIRÓ I ESTRUCH, IES "Abastos", Valencia, Spain; and the proposer.

M512. *Selected from a mathematics competition.*

A class of 20 students was given a three question quiz. Let x represent the number of students that answered the first question correctly. Similarly, let y and z represent the number of students that answered the second and the third questions correctly, respectively. If $x \geq y \geq z$ and $x + y + z \geq 40$, determine the smallest possible number of students who could have answered all three questions correctly in terms of x, y and z.
(*This questions was originally question 3c from the 2008 Galois Contest.*)

Solution by Florencio Cano Vargas, Inca, Spain.

The smallest number of students who have answered all three questions correctly corresponds to a situation where the number of students who have failed at least one question is maximum.

Taking into account that the number of students that failed question 1, 2 and 3 are $20-x$, $20-y$ and $20-z$ respectively, then the number of students who answered at least one question incorrectly is at most:

$$\max(20, 20 - x + 20 - y + 20 - z) = \max(20, 60 - x - y - z)\,.$$

Since $x + y + z \geq 40$ then

$$60 - x - y - z = 60 - (x + y + z) \leq 60 - 40 = 20\,.$$

And therefore the number of students who have failed at least one question is at most $60 - (x + y + z) \leq 20$. The number of students that answered all three questions correctly is at least:

$$20 - (60 - x - y - z) = x + y + z - 40\,.$$

Furthermore, since $x + y + z \leq 60$ we have the boundaries for the answer as:

$$0 \leq x + y + z - 40 \leq 20\,.$$

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.

THE CONTEST CORNER

No. 10

Shawn Godin

The Contest Corner *est une nouvelle rubrique offerte pas* ***Crux Mathematicorum****, comblant ainsi le vide suite à la mutation en 2013 de* Mathematical Mayhem *et* Skoliad *vers une nouvelle revue en ligne. Il s'agira d'un amalgame de* Skoliad*,* The Olympiad Corner *et l'ancien* Academy Corner *d'il y a plusieurs années. Les problèmes en vedette seront tirés de concours destinés aux écoles secondaires et au premier cycle universitaire ; les lecteurs seront invités à soumettre leurs solutions ; ces solutions commenceront à paraître au prochain numéro.*

Les solutions peuvent être envoyées à : Shawn Godin, Cairine Wilson S.S., 975 Orleans Blvd., Orleans, ON, CANADA, K1C 2Z5 *ou par couriel à* `crux-contest@cms.math.ca`.

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 avril 2014**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC46. Si on place l'entrée (m, n) dans la machine A, on obtient la sortie (n, m). Si on place l'entrée (m, n) dans la machine B, on obtient la sortie $(m + 3n, n)$. Si on place l'entrée (m, n) dans la machine C, on obtient la sortie $(m - 2n, n)$. Nathalie choisit le couple $(0, 1)$ et le place comme entrée dans une des machines. Elle prend ensuite la sortie et la place comme entrée dans n'importe quelle des machines. Elle continue de la sorte en prenant la sortie à chaque fois et en la plaçant comme entrée dans n'importe quelle des machines. (Par exemple, elle peut commencer par le couple $(0, 1)$ et utiliser successivement les machines B, B, A, C et B pour obtenir la sortie finale $(7, 6)$.) Est-il possible de commencer par le couple $(0, 1)$ et d'obtenir la sortie $(20132013, 20142014)$ en utilisant les machines dans n'importe quel ordre n'importe quel nombre de fois ?

CC47. Un cercle de rayon 2 est tangent aux deux côtés d'un angle. Un cercle de rayon 3 est tangent au premier cercle et aux deux côtés de l'angle. Un troisième cercle est tangent au deuxième cercle et aux deux côtés de l'angle. Déterminer le rayon du troisième cercle.

CC48. Déterminer s'il existe deux nombres réels a et b de manière que les deux polynômes $(x - a)^3 + (x - b)^2 + x$ et $(x - b)^3 + (x - a)^2 + x$ n'admettent que des zéros réels.

CC49. On a placé une pièce de monnaie sur certaines des 100 cases d'un quadrillage 10×10. Chaque case est à côté d'une case case qui contient une pièce de monnaie. Déterminer le nombre minimum de pièces de monnaie qui ont pu être placées. (On dit que deux cases distinctes sont à côté l'une de l'autre si elles ont un côté commun.)

CC50. On considère des entiers positifs de cinq chiffres ou moins. Démontrer que la racine carrée d'un tel entier ne peut avoir une approximation décimale qui commence par $0,1111$, mais qu'il existe un entier de huit chiffres dont l'approximation décimale commence par $0,1111$.

. . . .

CC46. Starting with the input (m, n), Machine A gives the output (n, m). Starting with the input (m, n), Machine B gives the output $(m + 3n, n)$. Starting with the input (m, n), Machine C gives the output $(m-2n, n)$. Natalie starts with the pair $(0, 1)$ and inputs it into one of the machines. She takes the output and inputs it into any one of the machines. She continues to take the output that she receives and inputs it into any one of the machines. (For example, starting with $(0, 1)$, she could use machines B, B, A, C, B in that order to obtain the output $(7, 6)$.) Is it possible for her to obtain $(20132013, 20142014)$ after repeating this process any number of times?

CC47. A circle of radius 2 is tangent to both sides of an angle. A circle of radius 3 is tangent to the first circle and both sides of the angle. A third circle is tangent to the second circle and both sides of the angle. Find the radius of the third circle.

CC48. Determine whether there exist two real numbers a and b such that both $(x - a)^3 + (x - b)^2 + x$ and $(x - b)^3 + (x - a)^2 + x$ contain only real roots.

CC49. Coins are placed on some of the 100 squares in a 10×10 grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal).

CC50. Show that the square root of a natural number of five or fewer digits never has a decimal part starting 0.1111, but that there is an eight-digit number with this property.

THE OLYMPIAD CORNER

No. 308

Nicolae Strungaru

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 avril 2014**.

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Rolland Gaudet, de l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC106. Déterminer tous les entiers positifs n pour lesquels tous les entiers à n positions décimales, contenant $n-1$ uns et 1 seul sept, sont premiers.

OC107. Le périmètre du triangle ABC est égal à 4. Des points X et Y sont placés sur les arcs AB et AC de façon à ce que $AX = AY = 1$. Les segments BC et XY s'entrecoupent au point M, à leur intérieur. Démontrer que le périmètre de l'un des triangles ABM et AMC est égal à 2.

OC108. Déterminer toutes les fonctions $f : \mathbb{R} \mapsto \mathbb{R}$ telles que

$$2f(x) = f(x+y) + f(x+2y)$$

pour tout $x \in \mathbb{R}$, $y \in [0, \infty)$.

OC109. Soit $a_1, a_2, \ldots, a_n, \ldots$ une permutation des entiers positifs. Démontrer qu'il existe infiniment d'entiers positifs i tels que $\text{pgcd}(a_i, a_{i+1}) \leq \frac{3}{4}i$.

OC110. Soit G un graphe qui ne contient pas K_4 comme sous graphe. Si le nombre de sommets de G est $3k$, où k est entier, quel est le nombre maximal de triangles dans G ?

. . . .

OC106. Find all the positive integers n for which all the n digit integers containing $n-1$ ones and 1 seven are prime.

OC107. The perimeter of triangle ABC is equal to 4. Points X and Y are marked on the rays AB and AC in such a way that $AX = AY = 1$. The segments BC and XY intersect at point M in their interior. Prove that the perimeter of one of the triangles ABM or AMC is equal to 2.

OC108. Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$2f(x) = f(x+y) + f(x+2y)$$

for all $x \in \mathbb{R}$, $y \in [0, \infty)$.

OC109. Let $a_1, a_2, \ldots, a_n, \ldots$ be a permutation of the set of positive integers. Prove that there exist infinitely many positive integers i such that $\gcd(a_i, a_{i+1}) \leq \frac{3}{4}i$.

OC110. Let G be a graph, not containing K_4 as a subgraph. If the number of vertices of G is $3k$, for some integer k, what is the maximum number of triangles in G?

OLYMPIAD SOLUTIONS

OC46. Let p be a prime number, and let x, y, z be integers so that $0 < x < y < z < p$. Suppose that x^3, y^3 and z^3 have the same remainders when divided by p. Prove that $x^2 + y^2 + z^2$ is divisible by $x + y + z$.
(*Originally question 5 from the 2009 Singapore Mathematical Olympiad, open section, round 2.*)

Solved by Michel Bataille, Rouen, France ; Chip Curtis, Missouri Southern State University, Joplin, MO, USA ; Oliver Geupel, Brühl, NRW, Germany ; David E. Manes, SUNY at Oneonta, Oneonta, NY, USA and Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA. We give the similar solutions of Bataille and Manes.

Let $x^3 \equiv y^3 \equiv z^3 \equiv a \pmod{p}$. Then the polynomial $P(w) = w^3 - a$ factors as

$$w^3 - a \equiv (w-x)(w-y)(w-z) \equiv w^3 - (x+y+z)w^2 + (xy+xz+yz)w - xyz \pmod{p} .$$

Thus

$$x + y + z \equiv xy + xz + yz \equiv 0 \pmod{p} .$$

Also

$$x^2 + y^2 + z^2 \equiv (x+y+z)^2 - 2(xy + xz + yz) \equiv 0 \pmod{p} .$$

As $0 < x+y+z < 3p$ we have $x+y+z = p$ or $x+y+z = 2p$. Then if $x+y+z = p$ the claim is obvious, while if $x + y + z = 2p$ it follows that $x^2 + y^2 + z^2$ is also even, and hence divisible by $2p = x + y + z$.

OC47. Let a, b be two distinct odd positive integers. Let a_n be the sequence defined as $a_1 = a$; $a_2 = b$; $a_n =$ the largest odd divisor of $a_{n-1} + a_{n-2}$. Prove that there exists a natural number N so that, for all $n \geq N$ we have $a_n = \gcd(a, b)$.
(*Originally question 7 from the 2009 India IMO selection test.*)

Solved by Arkady Alt, San Jose, CA, USA and Titu Zvonaru, Comăneşti, Romania. We give the solution of Zvonaru.

Let $d = \gcd(a, b)$. Since $a_{n-1} + a_{n-2} = 2^{\beta} a_n$, for some $\beta \geq 1$, we can easily deduce that $d \mid a_n$ for all n by induction. By induction it is also easy to show that $\gcd(a_n, a_{n-1}) = \gcd(a, b) = d$. Let $m = \max\{a, b\}$. Then for each n, as $a_{n-1} + a_{n-2}$ is even we have

$$a_n \leq \frac{a_{n-1} + a_{n-2}}{2} .$$

It follows immediately by induction that $a_n \leq m$ for all n.

Claim: If $a_k < a_{k+1}$ for some k then

$$a_n < a_{k+1} \text{ for all } n > k+1 .$$

We prove this by induction. Suppose that $a_k < a_{k+1}$ for some k then

$$a_{k+2} \leq \frac{a_{k+1} + a_k}{2} < \frac{a_{k+1} + a_{k+1}}{2} = a_{k+1}.$$

Similarly,

$$a_{k+3} \leq \frac{a_{k+2} + a_{k+1}}{2} < \frac{a_{k+1} + a_{k+1}}{2} = a_{k+1},$$

so the claim is true for $n = k+2$ and $n = k+3$. Suppose the claim is true for $n = j$ and $n = j+1$ for some $j \geq k+2$, then

$$a_{j+2} \leq \frac{a_{j+1} + a_j}{2} < \frac{a_{k+1} + a_{k+1}}{2} = a_{k+1} .$$

Thus the claim is true for all $n > k+1$.

It follows from here that there can only be finitely many k for which $a_k < a_{k+1}$. Indeed, assume by contradiction that we can find infinitely many $i_1 < i_2 < \cdots < i_n < \cdots$ so that

$$a_{i_j} < a_{i_j+1} .$$

Then, as $i_{j+1}+1 > i_j+1$ it follows from the claim that $a_{i_{j+1}+1} < a_{i_j+1}$ and hence

$$a_{i_1+1} > a_{i_2+1} > \cdots > a_{i_j+1} > \cdots$$

is a strictly decreasing infinite sequence of positive integers, contradiction.

Thus, there are only finitely many k for which $a_k < a_{k+1}$. Hence, there exists a largest such k. Thus, if k_0 denotes the largest such k, we have

$$a_n \geq a_{n+1} \text{ for all } n \geq k_0 + 1 .$$

As a_n is decreasing from $n = k$, and as it is positive, it cannot be strictly decreasing. Thus, there exists an $m \geq k$ so that $a_m = a_{m+1}$.

As $\gcd(a_m, a_{m+1}) = d$ we get that $a_m = a_{m+1} = d$, and then it is easy to prove by induction that $a_n = d$ for all $n \geq m$.

OC48. The angles of a triangle ABC are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. The bisectors meet the opposite sides at A', B' and C'. Prove the $A'B'C'$ is an isosceles triangle.
(*Originally question 2 from the 2009 Columbia Mathematical Olympiad.*)

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA and Titu Zvonaru, Comănești, Romania. We give the solution of Bataille.

We suppose $\angle BAC = \frac{4\pi}{7}, \angle ABC = \frac{2\pi}{7}, \angle BCA = \frac{\pi}{7}$ and we denote by a, b, c the side lengths BC, CA, AB respectively. We show that $A'B' = A'C'$.

The law of sines yields

$$\frac{a}{\sin\left(\frac{4\pi}{7}\right)} = \frac{b}{\sin\left(\frac{2\pi}{7}\right)} = \frac{c}{\sin\left(\frac{\pi}{7}\right)}.$$

Thus

$$\cos\left(\frac{2\pi}{7}\right) = \frac{\sin\left(\frac{4\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = \frac{a}{2b}.$$

Similarly

$$\cos\left(\frac{\pi}{7}\right) = \frac{b}{2c}.$$

From the bisector theorem we have

$$\frac{BC'}{a} = \frac{C'A}{b} = \frac{c}{a+b}; \quad \frac{BA'}{c} = \frac{CA'}{b} = \frac{a}{b+c}; \quad \frac{AB'}{c} = \frac{B'C}{a} = \frac{b}{c+a},$$

thus

$$BC' = \frac{ca}{a+b}; \quad BA' = \frac{ca}{b+c}; \quad CA' = \frac{ab}{b+c}; \quad CB' = \frac{ab}{c+a}.$$

Let I be the incentre of ABC. Then

$$\frac{AI}{AA'} = \frac{b+c}{a+b+c}; \qquad \frac{BI}{BB'} = \frac{a+c}{a+b+c}.$$

As $\angle ABA' = \frac{2\pi}{7} = \angle AB'B$ we get $IA = IB'$. Similarly $AA' = A'B$ and $BB' = B'C$. Thus

$$BI = \frac{a+c}{a+b+c}B'C = \frac{ab}{a+b+c}$$
$$IB' = IA = \frac{b+c}{a+b+c}AA' = \frac{c\,a}{a+b+c}$$

and adding these we get

$$\frac{ab}{a+c} = B'C = BB' = BI + IB' = \frac{a(b+c)}{a+b+c}.$$

Hence

$$b(a+b+c) = (a+c)(b+c) \Rightarrow b^2 = ac + c^2.$$

Also, $\cos(\frac{2\pi}{7}) = 2\cos^2(\frac{\pi}{7}) - 1$ yields

$$\frac{a}{2b} = \frac{b^2}{2c^2} - 1 \Rightarrow b^3 = ac^2 + 2bc^2\,.$$

Combining the last two relations we get

$$ab = ac + bc\,.$$

Then, we get

$$BC' = \frac{c^2}{b}; \quad BA' = \frac{bc}{a}; \quad CA' = \frac{b^2}{a}; \quad CB' = \frac{ac}{b}$$

and hence

$$\begin{aligned} A'C'^2 &= BA'^2 + BC'^2 - 2BA' \cdot BC' \cos\frac{2\pi}{7} = \frac{b^2c^2}{a^2} + \frac{c^4}{b^2} - \frac{c^3}{b}\ , \\ A'B'^2 &= CA'^2 + CB'^2 - 2CA' \cdot CB' \cos\frac{\pi}{7} = \frac{b^4}{a^2} - b^2 + \left(\frac{b^2 - c^2}{b}\right)^2 \\ &= \frac{b^4}{a^2} + \frac{c^4}{b^2} - 2c^2\ . \end{aligned}$$

Then,

$$\begin{aligned} A'C'^2 - A'B'^2 &= \frac{c}{ab}(b^3 - 2abc + ac^2) = \frac{c}{ab}(b^3 - 2(b^3 - bc^2) + ac^2) \\ &= \frac{c}{ab}(2bc^2 - b^3 + ac^2) = 0\,. \end{aligned}$$

Thus $A'B' = A'C'$ and we are done.
[*Ed.*: Covas mentioned that a proof can be found in Leon Bankoff and Jack Garfunkel, *The Heptagonal Triangle*, Mathematics Magazine 46 (1973), p. 17.]

OC49. Let N be a positive integer. How many non-congruent triangles are there, whose vertices lie on the vertices of a regular $6N$-gon?
(*Originally question 11 from the 2009 India IMO selection test.*)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

We claim there are $3N^2$ non-congruent triangles.

Draw the regular polygon on a circle of radius R so that the arc length between two consecutive vertices is 1. Then the circle has length $6N$. It is easy to see that two triangles with the vertices on the $6N$ points are congruent if and only if they have the same corresponding arc lengths on the circle. Thus, the number of non-congruent triangles is equal to the number of unordered partitions of $6N$ into 3 (not necessarily distinct) summands.

Thus, denoting by x, y, z the arc lengths, without loss of generality we have

$x \leq y \leq z$ and $x + y + z = 6N$. Thus, the number $g(N)$ of triangles becomes

$$\begin{aligned} g(N) &= \#\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\,,\ x + y \leq 6N - y\} \\ &= \#\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\,,\ x + 2y \leq 6N\} \\ &= \sum_{x=1}^{2N} \left(\left\lfloor \frac{6N - x}{2} \right\rfloor - (x-1) \right) \\ &= \sum_{\substack{x\,=\,1 \\ x \text{ is odd}}}^{2N-1} \left(3N - \frac{x+1}{2} - (x-1) \right) + \sum_{\substack{x\,=\,1 \\ x \text{ is even}}}^{2N} \left(3N - \frac{x}{2} - (x-1) \right) \\ &= \sum_{k=1}^{N} (3N - k - (2k-2)) + \sum_{k=1}^{N} (3N - k - (2k-1)) \\ &= \sum_{k=1}^{N} (6N - 6k + 3) \\ &= 6N^2 - 6\frac{N(N+1)}{2} + 3N = 6N^2 - 3N^2 - 3N + 3N = 3N^2 \end{aligned}$$

[*Ed.*: The stars and stripes method tell us that there are $\binom{6N-1}{3-1} = \frac{(6N-1)(6N-2)}{2}$ triples of (x, y, z) satisfying the equation. From those, 1 has $x = y = z$ and $3N - 2$ satisfy $x = y \neq z$ (the cases $x = y = 3N, z = 0$ and $x = y = z = 2N$ need to be eliminated). Thus by permutations, $9N - 6$ have exactly two values equal. The remaining $\frac{(6N-1)(6N-2)}{2} - 9N + 6 - 1$ have three distinct values. Counting now the unordered triples, the one with $(x = y = z)$ is one, the $9N - 6$ come in groups of three up to permutations, and the remaining $\frac{(6N-1)(6N-2)}{2} - 9N + 6 - 1$ come in groups of 6. Thus we have

$$\begin{aligned} &\frac{\frac{(6N-1)(6N-2)}{2} - 9N + 6 - 1}{6} + \frac{9N - 6}{3} + 1 \\ &\qquad = \frac{36N^2 - 18N + 2 - 18N + 10}{12} + 3N - 2 + 1 \\ &\qquad = 3N^2 - 3N + 1 + 3N - 1 = 3N^2\,. \end{aligned}$$
]

OC50. Let $n \geq 2$. If n divides $3^n + 4^n$, prove that 7 divides n.
(*Originally question 8 from the 2009 India IMO selection test.*)

Solved by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

As $3^n + 4^n$ is odd, it follows that n must be odd. Moreover, as $3^n + 4^n \equiv 1 \pmod 3$ it follows that $3 \nmid n$. Thus n is an odd integer greater than 3. Let p be the smallest prime dividing n.

Let

$$a \equiv 4 \cdot 3^{-1} \pmod p\,.$$

Then

$$0 \equiv 3^n + 4^n \equiv 3^n(1 + a^n) \pmod{p} .$$

As 3 is invertible mod p, we get

$$a^n \equiv -1 \pmod{p} .$$

As n is odd, this implies

$$(-a)^n \equiv 1 \pmod{p} .$$

Let r be the order of $-a$ modulo p. Then $r \mid n$ and $r \mid p - 1$. As p is the smallest divisor of n, $r < p$ and $r \mid n$, it follows that $r = 1$. Hence

$$1 \equiv -a \equiv -4 \cdot 3^{-1} \pmod{p}$$

and hence

$$3 \equiv -4 \pmod{p} .$$

Thus $p \mid 7$ and hence $p = 7$.

Unsolved Crux Problem

As remarked in the problem section, no problem is ever closed. We always accept new solutions and generalizations to past problems. Recently, Chris Fisher published a list of unsolved problems from *Crux* [2010 : 545, 547]. Below is one of these unsolved problems. Note that the solution to part (a) has been published [2005 : 468-470] but (b) remains open.

2977. [2004 : 429, 432; 2005 : 468-470] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers, let $r = \sqrt[n]{a_1 a_2 \cdots a_n}$, and let

$$E_n = \frac{1}{a_1(1+a_2)} + \frac{1}{a_2(1+a_3)} + \cdots + \frac{1}{a_n(1+a_1)} - \frac{n}{r(1+r)} .$$

(a) Prove that $E_n \geq 0$ for

- (a_1) $n = 3$;
- (a_2) $n = 4$ and $r \leq 1$;
- (a_3) $n = 5$ and $\frac{1}{2} \leq r \leq 2$;
- (a_4) $n = 6$ and $r = 1$.

(b)★ Prove or disprove that $E_n \geq 0$ for

- (b_1) $n = 5$ and $r > 0$;
- (b_2) $n = 6$ and $r \leq 1$.

BOOK REVIEWS

Amar Sodhi

Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry
by Glen Van Brummelen
Princeton University Press, 2012
ISBN: 978-0-69114-892-2, Hardcover, 192 + xvi pages, US$35
Reviewed by ***J. Chris Fisher****, University of Regina, Regina, SK*

The amusing title, playing on the two meanings of the word *heavenly*, sets the tone of the book — the author with his relaxed and playful style has produced a work that is as enjoyable as it is engrossing. Yet the book is quite serious in the goal suggested by its subtitle, *The Forgotten Art of Spherical Trigonometry.* With applications to navigation, astronomy, and geography, spherical trigonometry had been an essential part of every mathematician's education until the mid 1950s when the subject suddenly disappeared from North American schools. The thoroughness of its disappearance is evident in the pages of *Crux*, where geometry problems rarely deal with more than two dimensions, and when they do, they fail to attract much attention.

The book is not intended to be a thorough treatment of spherical trigonometry (among other things, the theorems are not presented in their full generality), nor is it a scholarly history of the subject. Instead it is a pleasant blend of history and mathematics. Each topic is introduced in a historical context. The problems that arise are then solved with the help of the basic theorems of spherical geometry, which are proved, for the most part, by adapting a historical proof. Van Brummelen declares early on that he appreciates the subject for its beauty; he finds the theorems elegant and often surprising; the proofs he provides have a visual impact that makes them informative and convincing.

Chapter 1 addresses the question of how to measure the earth's radius, which is accomplished by ascending a mountain in the fashion of al-Bīrūnī (around AD 1000), as opposed to the more familiar method of Eratosthenes (3rd century BC), which appears as an exercise. To complete the calculation one needs a sine table, which is constructed using ideas from Ptolemy (2nd century AD). The chapter ends with a calculation of the distance to the moon (also following Ptolemy). Chapter 2 provides an introduction to spherical geometry motivated by discussing the celestial sphere with its great circles representing the equator, ecliptic, and horizon. This leads to the question of how to describe the position of heavenly objects, the sun in particular, with respect to the celestial equator. Three answers are provided in the subsequent four chapters along with solutions to several other important problems: the ancient approach based on the spherical version of Menelaus's theorem, the medieval approach based on the rule of four quantities, and the modern approach based on Napier's discovery of how the parts of a spherical triangle are related.

These first six chapters form the basis for the courses and workshops that the author has taught. The text seems to be suitable for a university course for math and science majors, or for liberal arts students who are not afraid of mathematical formulas and proofs. The final three chapters evolved from student projects; they deal with areas, angles, polyhedra, stereographic projection, and navigating by the stars. Each chapter ends with numerous problems that illustrate and extend the material; many are taken from historical textbooks but, unfortunately, the author does not provide the solutions. The book can be read with profit from cover to cover by the casual reader armed only with the basics of plane trigonometry — very little is required beyond the geometric meaning of sine, cosine and tangent, and perhaps the sine and cosine laws. For the very casual reader whose interest is mainly in the pictures and anecdotes, the author has indicated the beginning and end of detailed arguments that, he says, can be omitted without losing the general flow of ideas. On the other hand, I found the mathematics fascinating; although I already knew much of the story I learned something, historical and mathematical, from every chapter.

Mathematical Excursions to the World's Great Buildings by Alexander J. Hahn
Princeton University Press, 2012
ISBN: 978-0-69114-520-4, Hardcover, 318 + ix pages, US$49.50
Reviewed by ***David Butt****, architect*
and ***J. Chris Fisher****, mathematician, Regina, SK*

By 1296 the Italian city of Florence was already becoming prosperous and important, and so it required a cathedral commensurate with its new status. Of course the town fathers wanted the latest Gothic vaulting, but to make their structure exceptional the plans called for the largest dome imaginable. As was customary in those days, construction of the nave and transept went on for over a century with no idea of how, or even if it might be possible to build such a dome — the octagonal drum on which the dome was to rest was 145 feet across, and it had to support a weight that was perhaps twenty times greater than that of any dome that had ever been built, yet obtrusive exterior buttresses were forbidden by the commission that oversaw the construction. Moreover, the drum itself reached 180 feet above the floor, a height which made infeasible the timber bracing built up from the ground that traditionally supported a dome as it was being constructed. An innovative solution was needed, and Filippo Brunelleschi (1377-1446), a brilliant mathematician, artist, craftsman, inventor, and architect had the skill to devise that solution.

Alexander Hahn lays a readable foundation for the breakthrough building projects we witness today. His analysis of historical structures should make every student of architecture proud. What better record do we have of man's creative evolution than our architecture? Architects and "master builders" rely heavily on the creative engineering skills of their design partners. The story of the building of the Florence Cathedral is one of a half-dozen compelling excursions to the world's great buildings. Perhaps even more exciting was the construction of the Sydney Opera House (1957-1973), which also required its designers to devise innovative solutions to numerous unprecedented problems; then, at the last minute, they had to bring in a bit of elementary mathematics to avert inundation by cost overruns.

The author has organized his book around two historical narratives: one explores the architectural form and structure of a sampling of great buildings; the other develops mathematics from a historical perspective. In addition to six stories that are recounted in some detail, there are numerous brief discussions, some quite interesting, but many only superficial. All, however, are accompanied by well-chosen pictures and diagrams. As for the mathematics in the book, the most successful deal with analysis of thrusts, loads, tensions, and compressions. Hahn uses modern mathematics (vectors, trigonometry, and calculus) to analyze structures whose builders had no access to such tools; the builders relied instead on experience, ingenuity, and faith. For example, the author is able to explain why cracks developed in Brunelleschi's magnificent dome, and to assure us that corrective measures are in place to secure the stability of the structure for centuries to come.

On the other hand, we suspect that readers of this journal would find most of the mathematics in the book to be boring; much of it falls far short of providing insight into the architecture. Large chunks of the book are devoted to elementary mathematics that has only a tenuous relationship to the main topic. It is not clear to us who the intended audience for those passages might be — perhaps the author used the book for a mathematics course that he taught, so he felt obliged to include more mathematics. He does not say. Architectural students would likely skip the mathematical digressions because they are distracting; readers with little background in mathematics would probably find the content too concise and not particularly well explained. In the chapter on Renaissance buildings, for example, the fifteen pages devoted to perspective drawings consist of calculations involving Cartesian coordinates that not only explain very little, but they are most certainly inappropriate for the job. After a messy three-page calculation to demonstrate that a circle becomes an ellipse in a perspective drawing, in place of the steps that determine the endpoint of the major axis he declares that, "This is an unpleasant computation that we will omit." After all that effort, we fail to learn how a Renaissance architect might have used drawings to design his building projects. It might be unusual for a review appearing in a mathematics journal to advise the reader to skip the mathematics, but following that advice, you will likely find Hahn's book fascinating reading.

PROBLEM SOLVER'S TOOLKIT

No. 3

Murray S. Klamkin

The Problem Solver's Toolkit is a new feature in ***Crux Mathematicorum****. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

On a Triangle Inequality

[*Ed. : Note, this article originally appeared in* ***Crux Mathematicorum*** *Volume 10, No. 5 [1984 : 139 - 140].*]

It is well known [4, p. 18] that, if A, B, C are the angles of a triangle, then

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}\,, \tag{1}$$

with equality if and only if $A = B = C$. (A proof is immediate from the concavity of the sine function on the interval $[0, \pi]$.) Vasić [1] generalized (1) to

$$x \sin A + y \sin B + z \sin C \le \frac{\sqrt{3}}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)\,, \tag{2}$$

where $x, y, z > 0$. In [2], the author showed that (2) was a special case of the two-triangle inequality

$$\begin{aligned} &4(xx' \sin A \sin A' + yy' \sin B \sin B' + zz' \sin C \sin C') \\ &\qquad \le \left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)\left(\frac{y'z'}{x'} + \frac{z'x'}{y'} + \frac{x'y'}{z'}\right)\,. \end{aligned} \tag{3}$$

Here we strengthen (2) to

$$x \sin A + y \sin B + z \sin C \le \frac{1}{2}(yz + zx + xy)\sqrt{\frac{x+y+z}{xyz}}\,. \tag{4}$$

We start with the polar moment of inertia inequality [3],

$$(w_1 + w_2 + w_3)(w_1R_1^2 + w_2R_2^2 + w_3R_3^2) \ge w_2w_3\alpha_1^2 + w_3w_1\alpha_2^2 + w_1w_2\alpha_3^2\,,$$

in which w_1, w_2, w_3 are arbitrary nonnegative numbers; $\alpha_1, \alpha_2, \alpha_3$ are the sides of a triangle $A_1A_2A_3$; and R_1, R_2, R_3 are the distances from an arbitrary point to the vertices of the triangle. Taking $R_1 = R_2 = R_3 = R$, the circumradius of the triangle, and using the power mean inequality,

$$\frac{w_2w_3\alpha_1^2 + w_3w_1\alpha_2^2 + w_1w_2\alpha_3^2}{w_2w_3 + w_3w_1 + w_1w_2} \geq \left\{\frac{w_2w_3\alpha_1 + w_3w_1\alpha_2 + w_1w_2\alpha_3}{w_2w_3 + w_3w_1 + w_1w_2}\right\}^2,$$

we obtain

$$R(w_1 + w_2 + w_3)\sqrt{w_2w_3 + w_3w_1 + w_1w_2} \geq w_2w_3\alpha_1 + w_3w_1\alpha_2 + w_1w_2\alpha_3 \,. \quad (5)$$

Now letting

$$w_1^2 = \frac{yz}{x}, \qquad w_2^2 = \frac{zx}{y}, \qquad w_3^2 = \frac{xy}{z},$$

and using $\alpha_i = 2R\sin A_i$ in (5), we obtain (4).

There is equality if and only if

$$\alpha_1 = \alpha_2 = \alpha_3$$

and the centroid of the weights w_1, w_2, w_3 at the respective vertices of the triangle coincides with the circumcentre. This entails that

$$w_1 = w_2 = w_3\,,$$

or, equivalently, that

$$x = y = z\,.$$

We have therefore shown that equality holds in (4) if and only if the triangle is equilateral.

Finally, to show that (4) is stronger than (2), we must establish that

$$\sqrt{3}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right) \geq (yz + zx + xy)\sqrt{\frac{x+y+z}{xyz}}$$

or, equivalently, that

$$3(y^2z^2 + z^2x^2 + x^2y^2)^2 \geq xyz(x+y+z)(yz+zx+xy)^2\,.$$

Letting $x = \frac{1}{u}$, $y = \frac{1}{v}$, and $z = \frac{1}{w}$ shows that this is equivalent to

$$3(u^2 + v^2 + w^2)^2 \geq (u+v+w)^2(vw + wu + uv)\,.$$

Since

$$\left(\frac{u^2+v^2+w^2}{3}\right)^2 \geq \left(\frac{u+v+w}{3}\right)^4$$

by the power mean inequality, it suffices finally to show that

$$\left(\frac{u+v+w}{3}\right)^2 \geq \frac{vw + wu + uv}{3},$$

and this is equivalent to

$$(v-w)^2 + (w-u)^2 + (u-v)^2 \geq 0\,.$$

References

[1] P. M. Vasić, "Some Inequalities for the Triangle", *Publ. Eletrotehn. Fak. Univ. Beogradu, Ser. Mat. Fiz.*, No. 274 - No. 301 (1964) 121-126.

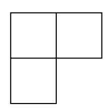
[2] M. S. Klamkin, "An Identity for Simplexes and Related Inequalties", *Simon Stevin*, 48 (1974-1975) 57-64.

[3] M. S. Klamkin, "Geometric Inequalities via the Polar Moment of Inertia", *Mathematics Magazine*, 48 (1975) 44-46.

[4] O. Bottema et al., *Geometric Inequalities*, Wolters-Noordhoff Publ., Groningen, 1969.

A Taste Of Mathematics
Aime-T-On les Mathématiques
ATOM

One square is deleted from a square "checkerboard" with 2^{2n} squares. Show that the remaining $2^{2n}-1$ squares can always be tiled with shapes of the form



which cover three squares.

This appears as problem 26 from *Mathematical Olympiads' Correspondence Program (1995-96)* by Edward J. Barbeau which is Volume I of the Canadian Mathematical Society's booklet series **ATOM**.

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RECURRING CRUX CONFIGURATIONS 9

J. Chris Fisher

Triangles Whose Angles Satisfy $B = 90° + C$

The main property of the triangles we look at this month first appeared in **Crux** in 2000:

Problem 2525 (April) [2000: 177; 2001: 270-271] (proposed by Antreas P. Hatzipolakis and Paul Yiu; reworded here). In a triangle ABC with $\angle B > \angle C$,

> $\angle B = 90° + \angle C$ *if and only if the centre of the nine-point circle lies on the line* BC.

The proposers also called for the reader to show that if, in addition, $\angle A = 60°$, then the 9-point centre N is the point where the bisector of angle A intersects BC. Of course, as we saw in Part 3 of this column, AN bisects angle A whenever $\angle A = 60°$, whatever B and C might be.

Problem 2765 [2002: 397; 2003: 349-351] (Proposed by K.R.S. Sastry). Derive a set of side-length expressions for the family of Heron triangles ABC in which the nine-point centre N lies on the line BC.

Michel Bataille's solution invoked Problem 2525; specifically, he proved that ΔABC has integer sides and area while $\angle B = 90° + \angle C$ if and only if there exists a primitive Pythagorean triple (m, n, k) with $m > n$ and a positive integer d for which

$$a = d(m^2 - n^2), \quad b = dkm, \quad c = dkn.$$

The area, given by Heron's fromula, equals $d^2mn(m^2 - n^2)/2$, which is an integer since m and n have opposite parity. The original problem required further that N lie in the interior of the segment BC; this will be the case if $m^2 > 3n^2$. For an explicit example having B between N and C we can set $m = 4, n = 3, k = 5$, and $d = 1$; the resulting triangle has $a = 7, b = 20, c = 15, \sin B = 4/5, \sin C = 3/5$, and area $= 42$.

As part of his solution, Sastry added a list of nine properties that are equivalent for any triangle ABC:

(1) The nine-point centre is on BC.

(2) $|B - C| = 90°$.

(3) $\tan B \tan C = -1$.

(4) $OA||BC$ (where O is the circumcentre of ΔABC).

(5) AH is tangent to the circumcircle of ΔABC (where H is the orthocentre).

(6) AH is tangent to the nine-point circle of ΔABC.

(7) BC bisects segment AH.

(8) $AN = \frac{1}{2}OH$.

(9) AC and BC trisect $\angle OCH$.

Problem 2867 [2003: 399; 2004: 378-379] (proposed by Antreas P. Hatzipolakis and Paul Yiu). With vertices B and C of triangle ABC fixed and its nine-point centre sliding along the line BC, the locus of the vertex A is the rectangular hyperbola (excluding B and C) whose major axis is BC.

When $B = C + 90°$ every point except B on the branch of the hyperbola through B is the vertex of such a triangle; when $C = B + 90°$, $A \neq C$ sweeps out the branch through C. Christopher J. Bradley remarked that this is the hyperbolic analogue of the familiar theorem about angles inscribed in semicircles: When BC is the diameter of a semicircle containing A, then $B + C = 90°$; when BC is the major axis of a rectangular hyperbola containing A, then $|B - C| = 90°$.

I conclude this month's essay, as well as this series, by recalling an observation made by Charles W. Trigg [1978: 79] that was included in Part 2 of the series: If the sides of ΔABC satisfy $c + a = 2b$ while the angles satisfy $A = C + 90°$, then the sides a, b, c are in the ratio $(\sqrt{7} + 1) : \sqrt{7} : \sqrt{7} - 1$.

Call for Problem of the Month Articles

We have introduced a few new features this volume. One of these new features is the *Problem of the Month* which is dedicated to the memory of former ***CRUX with MAYHEM*** Editor-in-Chief Jim Totten. The *Problem of the Month* features a problem and solution that we know Jim would have liked. Do you have a favourite problem that you would like to share? Write it up and send it to the editor at `crux-editors@cms.math.ca`. Articles featured in the *Problem of the Month* should be instructive, anecdotal and entertaining.

We are also seeking an editor for this column. If you are interested, or have someone to recommend, please contact the editor at the email address above.

PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1 juin 2014**. *Une étoile (⋆) après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5, 7, et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8, et 10, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier, de l'Université de Montréal, et Rolland Gaudet, de Université de Saint-Boniface, Winnipeg, MB, d'avoir traduit les problèmes.

3791. *Proposé par John Lander Leonard, Université de l'Arizona, Tucson, AZ.*

Les nombres de 0 à 12 sont répartis au hazard autour d'un cercle.

(a) Montrer qu'il doit exister un groupe de trois nombres adjacents dont la somme vaut au moins 18.

(b) Trouver le n maximal tel qu'il doit exister un groupe de trois nombres adjacents dont la somme vaut au moins n.

3792. *Proposé par Marcel Chiriţă, Bucharest, Romania.*

Résoudre le système suivant

$$2^x + 2^6 = 12$$
$$3^x + 3^y = 36$$

for $x, y \in \mathbb{R}$.

3793. *Proposé par George Apostolopoulos, Messolonghi, Grèce.*

Soit a, b et c trois nombres réels positifs tels que

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = 1007\sqrt{2}\ .$$

Trouver la valeur maximale de l'expression

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}\ .$$

3794. *Proposé par Václav Konečný, Big Rapids, MI, É-U.*

Soit ABC un triangle acutangle inscrit dans un cercle Γ, et soit A' et B' les pieds des hauteurs respectives issues de A et B. Soit respectivement P_A et P_B les deuxièmes intersections des cercles de diamètre BA' et CB' avec Γ. Déterminer l'angle entre les droites AP_A et BP_B.

3795. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b et c les longueurs des côtés d'un triangle ABC de hauteurs h_a, h_b, h_c et de rayon circonscrit R. Montrer que

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(R+\frac{h_a+h_b+h_c}{6}\right) > 3 \ .$$

3796. *Proposé par Michel Bataille, Rouen, France.*

Montrer que

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k)^{\binom{2n}{2k-1}}}{(2k+1)^{\binom{2n}{2k}}} = 1 \ .$$

3797. *Proposé par Panagiote Ligouras, École Secondaire Léonard de Vinci, Noci, Italie.*

Soit m_a, m_b et m_c les médianes et k_a, k_b et k_c les symédianes d'un triangle ABC. Si n est un entier positif, montrer que

$$\left(\frac{m_a}{k_a}\right)^n+\left(\frac{m_b}{k_b}\right)^n+\left(\frac{m_c}{k_c}\right)^n \geq 3 \ .$$

3798. *Proposé par Albert Stadler, Herrliberg, Suisse.*

Soit n un entier non négatif. Montrer que

$$\sum_{k=0}^{\infty} k^n\left(k+1-\frac{1}{k!}\int_1^{\infty} e^{-t}t^{k+1}dt\right) = \sum_{k=0}^{n}\frac{S(n,k)}{k+2},$$

où l'on pose $k^n = 1$ pour $k = n = 0$ et $S(n,k)$ sont les nombres de Stirling de seconde espèce, définis par la récurrence

$$S(n,m) = S(n-1,m-1) + mS(n-1,m), S(n,0) = \delta_{0,n}, S(n,n) = 1,$$

pour $n, m \geq 0$. A noter que $S(n,m) = 0$ quand $m > n$.
[*Ed* : A noter, ceci est la version corrigée du problème 3687.]

3799. *Proposé par Constantin Mateescu, "Zinca Golescu" Collège National, Pitesti, Roumanie.*

Soit respectivement R, r et s le rayon du cercle circonscrit, celui du cercle inscrit et le semi-périmètre d'un triangle ABC pour lequel on dénote $K = \sum_{\text{cyclique}} \sin\frac{A}{2}$.
Montrer que

$$s^2 = 4R(K-1)^2\left[R(K+1)^2+r\right] \ .$$

3800. *Proposé par Ovidiu Furdui, Campia Turzii, Cluj, Roumanie.*

Soit $n \geq 2$ un entier. Calculer

$$\int_0^\infty \int_0^\infty \left(\frac{e^{-x} - e^{-y}}{x - y} \right)^n dxdy \ .$$

. . . .

3791. *Proposed by John Lander Leonard, University of Arizona, Tucson, AZ.*

The numbers 0 through 12 are randomly arranged around a circle.

(a) Show that there must exist a trio of three adjacent numbers which sum to at least 18.

(b) Determine the maximum n such that there must exist a trio of three adjacent numbers which sum to at least n.

3792. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Solve the following system

$$2^x + 2^6 = 12$$
$$3^x + 3^y = 36$$

for $x, y \in \mathbb{R}$.

3793. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let a, b, and c be positive real numbers such that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = 1007\sqrt{2} \ .$$

Find the maximum value of the expression

$$\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \ .$$

3794. *Proposed by Václav Konečný, Big Rapids, MI, USA.*

Let an acute triangle ABC be inscribed in the circle Γ, and A' and B' be the feet of the altitudes from A and B respectively. Let the circle on diameter BA' intersect Γ again at P_A, and the circle on diameter CB' intersect Γ again at P_B. Determine the angle between the lines AP_A and BP_B.

3795. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c be the lengths of the sides of a triangle ABC with altitudes h_a, h_b, h_c and circumradius R. Prove that

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \left(R + \frac{h_a + h_b + h_c}{6} \right) > 3 \ .$$

3796. *Proposed by Michel Bataille, Rouen, France.*

Show that

$$\lim_{n\to\infty}\prod_{k=1}^{n}\frac{(2k)^{\binom{2n}{2k-1}}}{(2k+1)^{\binom{2n}{2k}}}=1\ .$$

3797. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let m_a, m_b, m_c be the medians and k_a, k_b, k_c be the symmedians of a triangle ABC. If n is a positive integer, prove that

$$\left(\frac{m_a}{k_a}\right)^n+\left(\frac{m_b}{k_b}\right)^n+\left(\frac{m_c}{k_c}\right)^n\geq 3\ .$$

3798. *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{\infty}k^n\left(k+1-\frac{1}{k!}\int_1^{\infty}e^{-t}t^{k+1}dt\right)=\sum_{k=0}^{n}\frac{S(n,k)}{k+2},$$

where k^n is taken to be 1 for $k=n=0$ and $S(n,k)$ are the Stirling numbers of the second kind that are defined by the recursion

$$S(n,m)=S(n-1,m-1)+mS(n-1,m),S(n,0)=\delta_{0,n},S(n,n)=1,$$

for $n,m\geq 0$. Note also that $S(n,m)=0$ when $m>n$.
[*Ed: Note, this is the corrected version of problem 3687.*]

3799. *Proposed by Constantin Mateescu, "Zinca Golescu" National College, Pitesti, Romania.*

Let ABC be a triangle with circumradius R, inradius r and semiperimeter s for which we denote $K=\sum_{\text{cyclic}}\sin\frac{A}{2}$. Prove that

$$s^2=4R(K-1)^2\left[R(K+1)^2+r\right]\ .$$

3800. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $n\geq 2$ be an integer. Calculate

$$\int_0^{\infty}\int_0^{\infty}\left(\frac{e^{-x}-e^{-y}}{x-y}\right)^n dxdy\ .$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3691. [2011 : 541, 543] *Proposed by Hung Pham Kim, student, Stanford University, Palo Alto, CA, USA.*

Let a, b, and c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab} \;\leq\; 1\,.$$

Solution by Oliver Geupel, Brühl, NRW, Germany.

Since for nonnegative real numbers such that $a+b+c=3$ we have

$$a^2b+b^2c+c^2a+abc\leq 4 \tag{1}$$

(See the lemma in the solution of Problem 3549 [2011:253].), then

$$\begin{aligned}4&\left(\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab}-1\right)\\ &= a^2b\left(\frac{bc}{4-bc}+1\right)+b^2c\left(\frac{ca}{4-ca}+1\right)+c^2a\left(\frac{ab}{4-ab}+1\right)-4\\ &\leq \frac{a^2b^2c}{4-bc}+\frac{b^2c^2a}{4-ca}+\frac{c^2a^2b}{4-ab}-abc.\end{aligned}$$

So it suffices to prove that

$$\frac{ab}{4-bc}+\frac{bc}{4-ca}+\frac{ca}{4-ab}-1\leq 0.$$

Clearing denominators, it becomes

$$\begin{aligned}&32(ab+bc+ca)+abc(a^2b+b^2c+c^2a+abc)\\ &\qquad -64-8(a+b+c)abc-4(a^2b^2+b^2c^2+c^2a^2)\leq 0.\end{aligned}$$

After applying inequality (1) and homogenizing, the inequality can be written in the form

$$\begin{aligned}&\frac{32}{9}(a+b+c)^2(ab+bc+ca)+\frac{4}{3}(a+b+c)abc\\ &\qquad -\frac{64}{81}(a+b+c)^4-8(a+b+c)abc-4(a^2b^2+b^2c^2+c^2a^2)\leq 0.\end{aligned}$$

Clearing denominators again, expanding and adopting the notation

$$[\alpha,\beta,\gamma]=\sum_{sym}a^\alpha b^\beta c^\gamma,$$

it becomes $16([3,1,0]-[4,0,0])+33([2,1,1]-[2,2,0]) \leq 0$, which is true by Muirhead's theorem. This completes the proof.

Notice that the equality holds for $a=b=c=1$, or $a=2, b=1, c=0$, or any permutations of these values.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

The proposer posted this problem in 2008, on the Mathlinks forum, see http://www.artofproblemsolving.com/Forum/viewtopic.php?t=189307. *Also, you can find a nice solution by Vo Quoc Ba Can on* http://canhang2007.wordpress.com/2009/11/02/inequality-37-p-k-hung.

3692. [2011 : 541, 543] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

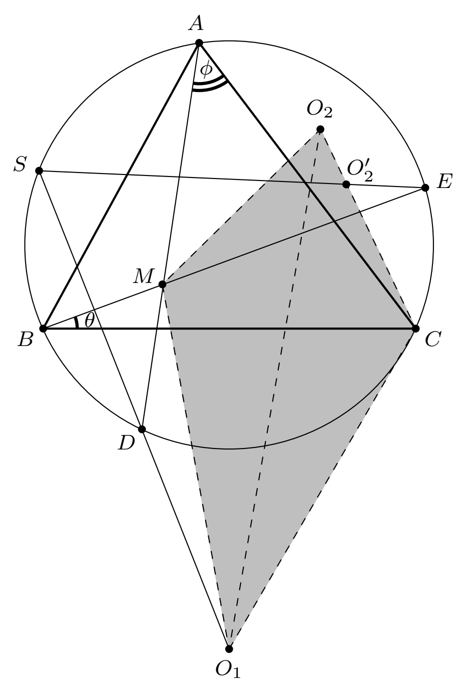
For an arbitrary point M in the plane of triangle ABC define D, E, and F to be the second points where the circumcircle meets the lines AM, BM, and CM, respectively. If O_1, O_2, and O_3 are the respective centres of the circles BCM, CAM, and ABM, prove that DO_1, EO_2, and FO_3 are concurrent.

Identical solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina and Salem Malikić, student, Simon Fraser University, Burnaby, BC, with the notation modified by the editor.

Editor's comment. To avoid the need for numerous special cases, we shall make use of directed angles: the symbol $\angle PQR$ will represent the angle through which the line QP must be rotated about Q to coincide with QR, where $0^\circ \leq \angle PQR < 180^\circ$. Properties of directed angles are discussed in Roger Johnson's *Advanced Euclidean Geometry*, Dover reprint (1960).

Our goal is to prove that not only are DO_1, EO_2, FO_3 concurrent, but their common point lies on the circumcircle of ΔABC. Denote by S the second point where the line DO_1 intersects the circumcircle, and let $\phi = \angle MAC = \angle DAC$ and $\theta = \angle CBM = \angle CBE$. Because A, C, D, E and S are all on the same circle we have

$$\begin{aligned}\angle O_1SE &= \angle DSE \\ &= \angle DSC + \angle CSE \\ &= \angle DAC + \angle CBE \\ &= \phi + \theta. \qquad (1)\end{aligned}$$



Since O_1 and O_2 are circumcentres of triangles MBC and MAC, respectively, we have

$$\angle MO_2C = 2\angle MAC = 2\phi$$

and

$$\angle CO_1M = 2\angle CBM = 2\theta.$$

Moreover, the radii $O_2M = O_2C$ and $O_1M = O_1C$, so that the quadrilateral MO_2CO_1 is a kite, implying that

$$\angle O_1O_2C = \frac{1}{2}\angle MO_2C = \phi \text{ and } \angle CO_1O_2 = \frac{1}{2}\angle CO_1M = \theta. \qquad (2)$$

Consequently, in ΔO_1CO_2 we have

$$\angle O_1CO_2 = \phi + \theta. \qquad (3)$$

We denote by O_2' the point where ES intersects CO_2 and will prove that in fact, $O_2' = O_2$. From (1) and (3) we have

$$\angle O_1CO_2' = \angle O_1CO_2 = \phi + \theta = \angle O_1SE = \angle O_1SO_2'.$$

Consequently the points O_1, C, O_2', S are concyclic, whence (with the help of (2))

$$\angle O_1O_2'C = \angle O_1SC = \angle DSC = \angle DAC = \phi = \angle O_1O_2C.$$

Because there is exactly one line through O_1, namely O_1O_2, that makes a *directed* angle of ϕ with the line CO_2, we conclude that $O_2' = O_2$; that is, EO_2 passes through S, as claimed. Interchanging the roles of B and C, one shows similarly that FO_3 also passes through S.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France (2 solutions); and the proposer.

Apostolopoulos provided a neat computation, using complex numbers to show that the point where DO_1 intersects the circumcircle is represented by an expression that is symmetric in A, B, and C.

3693. [2011 : 541, 543] *Proposed by Michel Bataille, Rouen, France.*

Given $k \in \left(\frac{1}{4}, 0\right)$, let $\{a_n\}_{n=0}^{\infty}$ be the sequence defined by $a_0 = 2$, $a_1 = 1$ and the recursion $a_{n+2} = a_{n+1} + ka_n$. Evaluate

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2}.$$

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Using roots of the characteristic equation, $x^2 - x - k = 0$, of the given recursion, we have

$$a_n = \left(\frac{1+\sqrt{1+4k}}{2}\right)^n + \left(\frac{1-\sqrt{1+4k}}{2}\right)^n, \qquad k \in (-1/4, 0).$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\left(\frac{1+\sqrt{1+4k}}{2} \right)^n + \left(\frac{1-\sqrt{1+4k}}{2} \right)^n \right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^2} x^n + \sum_{n=1}^{\infty} \frac{1}{n^2} (1-x)^n := S(x),$$

where $x = \frac{1+\sqrt{1+4k}}{2}$. Since

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = \sum_{n=1}^{\infty} \int_0^x t^{n-1} dt = \int_0^x \sum_{n=1}^{\infty} t^{n-1} dt = \int_0^x \frac{1}{1-t} dt = -\ln(1-x)$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} (1-x)^n = -\ln x,$$

then

$$S'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} + \sum_{n=1}^{\infty} \frac{(1-x)^{n-1}}{n}$$
$$= -\frac{1}{n} \ln(1-x) + \frac{1}{1-x} \ln x$$
$$= -(\ln x \cdot \ln(1-x))'.$$

Hence

$$S(x) = -\ln x \cdot \ln(1-x) + C.$$

Since $\lim_{x\to 0^+} \ln x \cdot \ln(1-x) = 0$ and $S(0) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, then $C = \frac{\pi^2}{6}$, and we have the final result

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} = \frac{\pi^2}{6} - \ln \frac{1+\sqrt{1+4k}}{2} \cdot \ln \frac{1-\sqrt{1+4k}}{2}.$$

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RADOUAN BOUKHARFANE, Polytechnique de Montréal, Montréal, PQ; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; ANASTASIOS KOTRONIS, Athens, Greece; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; M. A. PRASAD, India; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and YANPING XIA, Southeast Missouri State University, Cape Girardeau, MO, USA; and the proposer.

3694. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let x, y, and z be nonnegative real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$\sqrt{1-\left(\frac{x+y}{2}\right)^2}+\sqrt{1-\left(\frac{y+z}{2}\right)^2}+\sqrt{1-\left(\frac{x+z}{2}\right)^2} \ \geq\ \sqrt{6}\,.$$

[*Editor's note*: Both AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and Albert Stadler, Herrliberg, Switzerland, pointed out that the same problem by the same proposer had appeared as Example 1 of the article "Square it" published in Vol. 12, No. 5 (2008) of *Mathematical Excalibur* (`http://www.math.ust.hk/excalibur/v12_n5.pdf`). The solution given there, which used the Cauchy-Schwarz Inequality, is similar to most of the submitted solutions we received.]

Solved by ARKADY ALT, San Jose, CA, USA; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL BRACKEN, University of Texas, Edinburg, TX, USA; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; M. A. PRASAD, India; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

3695. [2011 : 541, 544] *Proposed by Michel Bataille, Rouen, France.*

Let a be a positive real number. Find all strictly monotone functions $f:(0,\infty)\to(0,\infty)$ such that

$$(x+a)f(x+y) = af(yf(x))$$

for all positive x,y.

I. Solution by Oliver Geupel, Brühl, NRW, Germany.

It is straightforward to check that $f(x)=\frac{a}{x+a}$ is a solution. We prove that it is unique.

Suppose that f meets the requirements of the problem. Since f is monotone, it has a limit L (possibly infinite) from the right at 0. For each $x>0$, we have that

$$\lim_{y\to 0^+} f(x+y) = \frac{a}{x+a}\lim_{y\to 0^+} f(yf(x)) = \frac{a}{x+a}L.$$

Thus the limit L must be a positive real number. Since a monotone function has at most countably many discontinuities, $f(x)=(aL)/(x+a)$ on a dense subset of $(0,\infty)$ and so $f(x)$ must be decreasing.

For any $x>0$ and each $\epsilon>0$, we can choose positive numbers x_1 and x_2 such that $x-\epsilon<x_1<x<x_2<x+\epsilon$ and

$$\frac{a}{x_1+a}\cdot L = f(x_1)\geq f(x)\geq f(x_2)=\frac{a}{x_2+a}\cdot L.$$

Therefore, we must have that

$$f(x)=\frac{aL}{x+a}$$

for all $x>0$. Substituting this expression into the functional equation reveals that L must be 1. Note that we did not require the monotonicity to be strict.

II. Solution by M. A. Prasad, India.

Define $g(x)=f(ax)$ for $x>0$. We get that

$$(x+1)g(x+y)=g(yg(x))$$

for $x,y>0$. For sufficiently large values of y, we may write

$$\begin{aligned}(x+1)g(x+y)=g(1+(yg(x)-1))&=\frac{1}{2}g((yg(x)-1)g(1))\\&=\frac{1}{2}g\left(\frac{(yg(x)-1)g(1)}{g(2x+1)}g(2x+1)\right)\\&=\frac{1}{2}(2x+2)g\left(2x+1+\frac{yg(x)-1}{g(2x+1)}g(1)\right).\end{aligned}$$

Since the function is strictly monotone, it is one-one, so that

$$x+y=2x+1+\frac{yg(x)-1}{g(2x+1)}g(1).$$

For each x, this holds for infinitely many values of y, so that, equating terms independent of y and coefficients of y, we get that

$$x=2x+1-\frac{g(1)}{g(2x+1)}\Leftrightarrow(x+1)g(2x+1)=g(1).$$

and

$$g(2x+1)=g(x)g(1).$$

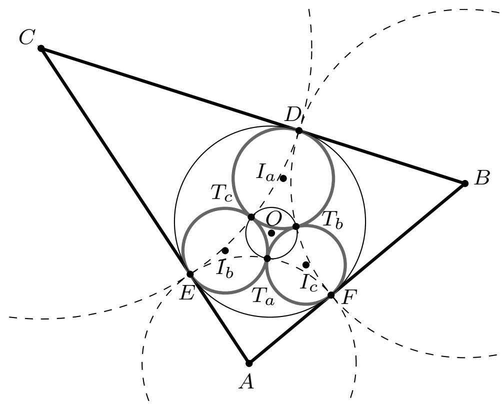
Eliminating $g(2x+1)$ from these equations yields that $(x+1)g(x)=1$, from which $f(x)$ can be obtained.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; and the proposer.

3696. [2011 : 541, 544] *Proposed by Nguyen Thanh Binh, Hanoi, Vietnam.*

Let the incircle of triangle ABC touch the sides BC at D, CA at E, and AB at F. Construct by ruler and compass the three mutually tangent circles that are internally tangent to the incircle, one at D, one at E, and one at F.

Solution by George Apostolopoulos, Messolonghi, Greece.



We will use notation $\odot A$ to represent a circle with centre at A.
Let $\odot A$, $\odot B$, and $\odot C$ be the circles with radii AE, BF, and CD, respectively. First, we will construct a small circle $\odot O$, externally tangent to all of $\odot A$, $\odot B$, and $\odot C$.
(This is not a difficult but rather cumbersome construction and it can be found for example on the website:
http://oz.nthu.edu.tw/~g9721504/soddycircles.html .)
If T_a, T_b, T_c denote the corresponding tangent points, as on the diagram, then the circle $\odot I_c$ inscribed in $\triangle ABO$ is the circumcircle of $\triangle FT_aT_b$. Similarly, let $\odot I_a$ and $\odot I_b$ be the inscribed circles in $\triangle BCO$ and $\triangle CAO$, respectively.
We claim that the circles $\odot I_a$, $\odot I_b$, $\odot I_c$ satisfy conditions of the problem.
Firstly, they are tangent to the lines AB, BC, and CA at points E, D, and F, respectively, which means that they are tangent internally to the incircle of $\triangle ABC$ at the corresponding points E, D, and F.
Secondly, $\odot I_a$ and $\odot I_b$ are both tangent to the line CO at the common point T_c, hence they are externally tangent to each other. Since similar conclusion refers to the remaining pairs of circles, the claim holds.

Also solved by MICHEL BATAILLE, Rouen, France; OLIVER GEUPEL, Brühl, NRW, Germany; and the proposer.

3697. [2011 : 542, 544] *Proposed by Michel Bataille, Rouen, France.*

For positive integer n, prove that

$$\left(\tan\frac{\pi}{7}\right)^{6n}+\left(\tan\frac{2\pi}{7}\right)^{6n}+\left(\tan\frac{3\pi}{7}\right)^{6n}$$

is an integer and find the highest power of 7 dividing this integer.

Solution by Roy Barbara, Lebanese University, Fanar, Lebanon, expanded slightly by the editor.

We prove more generally that $f(n)=\left(\tan\frac{\pi}{7}\right)^{2n}+\left(\tan\frac{2\pi}{7}\right)^{2n}+\left(\tan\frac{3\pi}{7}\right)^{2n}$ is an integer for all $n\in\mathbb{N}$.

Let $a=\left(\tan\frac{\pi}{7}\right)^{2}$, $b=\left(\tan\frac{2\pi}{7}\right)^{2}$ and $c=\left(\tan\frac{3\pi}{7}\right)^{2}$. Then $f(n)=a^n+b^n+c^n$.

It is well known (see [1]) that $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$, and $\tan\frac{3\pi}{7}$ are zeros of the polynomial $x^6-21x^4+35x^2-7$. Therefore a, b, and c are three (distinct) roots of $x^3-21x^2+35x-7$.

It follows that $a+b+c=21$, $ab+bc+ca=35$ and $abc=7$. Hence

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=21^2-70=371,$$

and

$$a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-bc-ca+3abc=21(371-35)+21=7077.$$

Since

$$\begin{aligned}a^{n+3}+b^{n+3}+c^{n+3}&=(a+b+c)(a^{n+2}+b^{n+2}+c^{n+2})\\&\quad-(ab+bc+ca)(a^{n+1}+b^{n+1}+c^{n+1})+abc(a^n+b^n+c^n)\end{aligned}$$

we get

$$f(n+3)=21f(n+2)-35f(n+1)+7f(n) \tag{1}$$

and since $f(1)=21$, $f(2)=371$, and $f(3)=7077$ are all integers, it follows from (1) that $f(n)$ is an integer for all $n\in\mathbb{N}$.

We now show that $7^n \parallel f(3n)$: that is, the highest power of 7 dividing $f(3n)$ is n.

Since $f(1)$, $f(2)$, and $f(3)$ are all multiples of 7, we have $7\mid f(k)$ for $k=1$, 2, and 3. Suppose that $7^k\mid f(3k-2)$, $7^k\mid f(3k-1)$ and $7^k\mid f(3k)$ for all $k=1,2,\ldots,n$ for some $n\geq 1$.

Then replacing n in (1) by $3n-2$, $3n-1$, and $3n$, respectively, we get

$$f(3n+1)=21f(3n)-35f(3n-1)+7f(3n-2) \tag{2}$$
$$f(3n+2)=21f(3n+1)-35f(3n)+7f(3n-1) \tag{3}$$
$$f(3n+3)=21f(3n+2)-35f(3n+1)+7f(3n). \tag{4}$$

Using (2) – (4) we obtain successively that $7^{n+1}\mid f(3n+1)$, $7^{n+1}\mid f(3n+2)$, and $7^{n+1}\mid f(3n+3)$. That is, $7^{n+1}\mid f(3(n+1))$. Hence by induction we conclude that $7^n\mid f(3n)$ for all $n\in\mathbb{N}$.

To complete the proof, it remains to show that $7^{n+1}\nmid f(3n)$. To this end, we use induction to prove that

$$\frac{f(3n)}{7^n}\equiv 3 \pmod 7\,. \tag{5}$$

Since $f(3)=7077=7\times 1011$ and $1011\equiv 3 \pmod 7$, (5) is true for $n=1$.

Suppose (5) holds for some $n\geq 1$. We let $f(3n)=7^nq$ where $q\equiv 3 \pmod 7$. We also set $f(3n+1)=7^{n+1}r$, $f(3n+2)=7^{n+1}s$, and $f(3n+3)=7^{n+1}t$, where r, s, and t are integers. Then by (4) we obtain $7^{n+1}t=21(7^{n+1}s)-35(7^{n+1}r)+7(7^nq)$ which, upon dividing by 7^{n+1}, yields $t=7(3s)-7(5r)+q\equiv q\equiv 3 \pmod 7$. That is $\frac{f(3n+3)}{7^{n+1}}\equiv 3 \pmod 7$ and the induction is complete.

References

[1] mathworld.wolfram.com/TrigonometryAnglesPi7.html.

Also solved by AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

3698. [2011 : 542, 544] *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Find the value of

$$\lim_{n\to\infty}\int_0^1 \sqrt[n]{1+n^n x^{n^2}}\,dx\,.$$

I. Solution by Albert Stadler, Herrliberg, Switzerland.

Let

$$I_n=\int_0^{1/\sqrt[n]{n}} \sqrt[n]{1+n^n x^{n^2}}\,dx,\quad \text{and}\quad J_n=\int_{1/\sqrt[n]{n}}^1 \sqrt[n]{1+n^n x^{n^2}}\,dx.$$

Then

$$\frac{1}{\sqrt[n]{n}}=\int_0^{1/\sqrt[n]{n}} dx \le I_n \le \int_0^{1/\sqrt[n]{n}} \sqrt[n]{2}\,dx \le \sqrt[n]{2},$$

and

$$\int_{1/\sqrt[n]{n}}^1 \sqrt[n]{n^n x^{n^2}}\,dx \le J_n \le \int_{1/\sqrt[n]{n}}^1 \sqrt[n]{2n^n x^{n^2}}\,dx.$$

Since

$$\int_{1/\sqrt[n]{n}}^1 \sqrt[n]{n^n x^{n^2}}\,dx = n\int_{1/\sqrt[n]{n}}^1 x^n\,dx=\frac{1}{n+1}\left(n-n^{-1/n}\right),$$

we find that $\lim_{n\to\infty} I_n=\lim_{n\to\infty} J_n=1$, so that the required limit is equal to 2.

II. Solution by George Apostolopoulos, Messolonghi, Greece; Moubinool Omarjee, Lycée Henri IV, Paris, France; and M. A. Prasad, India(independently).

Since $\sqrt[n]{1+n^n x^{n^2}} < 1+nx^n$ for $0<x<1$. Then

$$\int_0^1 \sqrt[n]{1+n^n x^{n+1}}\,dx < \int_0^1 (1+nx^n)\,dx = 1+\frac{n}{n+1}.$$

On the other hand, when $0<c<1$, we have that

$$\int_0^1 \sqrt[n]{1+n^n x^{n^2}}\,dx=\int_0^c \sqrt[n]{1+n^n x^{n^2}}\,dx+\int_c^1 \sqrt[n]{1+n^n x^{n^2}}\,dx$$
$$> \int_0^c 1\,dx+\int_c^1 nx^n\,dx = c+\frac{n}{n+1}(1-c^{n+1}).$$

Therefore the required limit is not less than $c+1$ for each $c \in (0,1)$. It follows that the limit is 2.

Also solved by PAUL BRACKEN, University of Texas, Edinburg, TX, USA; KEE-WAI LAU, Hong Kong, China; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. In the second solution, Omarjee avoided the extra parameter by taking the specific value $c = n^{-1/n}$. *Perfetti and the proposer made a variable transformation* $y = nx^n$ *that rendered the integral as the sum of two integrals over* $[0,1]$ *and* $[1,n]$, *each of which tended to 1. Lau made a transformation* $y = n^{1/n}x$. *Two incorrect solutions were received.*

3699. [2011 : 542, 544] *Proposed by Mehmet Şahin, Ankara, Turkey.*

Let ABC denote a triangle, I its incenter, and ρ_a, ρ_b, and ρ_c the inradii of IBC, ICA, and IAB, respectively. Prove that

$$\frac{1}{\rho_a} + \frac{1}{\rho_b} + \frac{1}{\rho_c} \ \geq \ \frac{18\tan(75^\circ)}{a+b+c}\,.$$

I. Solution by George Apostolopoulos, Messolonghi, Greece.

Let $2\alpha = B+C$, $2\beta = C+A$ and $2\gamma = A+B$. Then $\alpha+\beta+\gamma = 180^\circ$, $a = r(\tan\beta + \tan\gamma)$, $b = r(\tan\gamma + \tan\alpha)$, $c = r(\tan\alpha + \tan\beta)$, $IB = r\sec\beta$, $IC = r\sec\gamma$ and $2[IBC] = ar$ and $2[IBC] = ar$, so that

$$\frac{1}{\rho_a} = \frac{1}{r} + \frac{\cos\beta + \cos\gamma}{r\sin\alpha}$$

with analogous expressions for $1/\rho_b$ and $1/\rho_c$.

Observe that $\tan 75^\circ = \sqrt{3}+2$ and $\tan\alpha\cdot\tan\beta\cdot\tan\gamma = \tan\alpha+\tan\beta+\tan\gamma \geq 3\tan 60^\circ = 3\sqrt{3}$ (from the convexity of the tangent function). Then

$$\frac{18\tan 75^\circ}{a+b+c} = \frac{9\sqrt{3}+18}{r\tan\alpha\cdot\tan\beta\cdot\tan\gamma}.$$

Thus, it is required to show that

$$3\tan\alpha\cdot\tan\beta\cdot\tan\gamma + \sum_{\text{cyclic}}\left(\frac{\cos\beta+\cos\gamma}{\cos\alpha}\right)\tan\beta\cdot\tan\gamma \geq 9\sqrt{3}+18.$$

This inequality holds for the two terms respectively, that for the second term relying on an application of the Arithmetic-Geometric Means inequality.

II. Solution by Oliver Geupel, Brühl, NRW, Germany.

We note that $\tan 75^\circ = \tan(45^\circ + 30^\circ) = 2+\sqrt{3}$. Let r, R and s denote the inradius, circumradius and semiperimeter, respectively, of triangle ABC. Let h_I, h_A, h_B and $[IAB]$ denote the altitudes and area of triangle IAB. We have that $h_I = r$, $h_A = c\sin(B/2)$ and $h_B = c\sin(A/2)$. Hence

$$\frac{1}{\rho_a} = \frac{AB+AI+BI}{2[IAB]} = \frac{1}{h_I} + \frac{1}{h_B} + \frac{1}{h_A} = \frac{1}{r} + \frac{1}{c\sin(A/2)} + \frac{1}{c\sin(B/2)}.$$

Similar identities hold for $1/\rho_b$ and $1/\rho_c$. We find that

$$\begin{aligned}(a+b+c)&\left(\frac{1}{\rho_a}+\frac{1}{\rho_b}+\frac{1}{\rho_c}\right)\\ &=\frac{3(a+b+c)}{r}+(a+b+c)\left(\frac{1}{a\sin(B/2)}+\frac{1}{b\sin(C/2)}+\frac{1}{c\sin(A/2)}\right)\\ &+(a+b+c)\left(\frac{1}{a\sin(C/2)}+\frac{1}{b\sin(A/2)}+\frac{1}{c\sin(B/2)}\right).\end{aligned}$$

By the Arithmetic-Geometric Means inequality, we have that

$$\begin{aligned}\frac{3(a+b+c)}{r}&=\frac{6s}{\sqrt{(s-a)(s-b)(s-c)/s}}\\ &\geq\frac{6s^{3/2}}{(s/3)^{3/2}}=6\cdot 3^{3/2}=18\sqrt{3}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, the Arithmetic-Geometric Means inequality and Euler's inequality $R\geq 2r$ in succession, we obtain that

$$\begin{aligned}(a+b+c)&\left(\frac{1}{a\sin(B/2)}+\frac{1}{b\sin(C/2)}+\frac{1}{c\sin(A/2)}\right)\\ &\geq\left(\frac{1}{\sqrt{\sin(A/2)}}+\frac{1}{\sqrt{\sin(B/2)}}+\frac{1}{\sqrt{\sin(C/2)}}\right)^2\\ &\geq\frac{9}{\sqrt[3]{\sin(A/2)\sin(B/2)\sin(C/2)}}=9\sqrt[3]{\frac{4R}{r}}\geq 18.\end{aligned}$$

Analogously,

$$(a+b+c)\left(\frac{1}{a\sin(C/2)}+\frac{1}{b\sin(A/2)}+\frac{1}{c\sin(B/2)}\right)\geq 18.$$

Consequently,

$$(a+b+c)\left(\frac{1}{\rho_a}+\frac{1}{\rho_b}+\frac{1}{\rho_c}\right)\geq 18(2+\sqrt{3})=18\tan 75^\circ.$$

Equality holds if and only if triangle ABC is equilateral.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; NECULAI STANCIU, George Emil Palade Secondary School, Buzău, Romania and TITU ZVONARU, Comănești, Romania (joint); and the proposer.

3700. [2011 : 542, 544] *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle and $a = BC$, $b = CA$, $c = AB$. Given that

$$aPA^2 + cPB^2 + bPC^2 = cPA^2 + bPB^2 + aPC^2 = bPA^2 + aPB^2 + cPC^2$$

for some point P, show that $\triangle ABC$ is equilateral.

Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.

The problem is missing a necessary restriction; the correct statement (extended to three-dimensional space by the editors): *Let ℓ be the line in Euclidean space that is that is perpendicular to the plane of the given triangle ABC and passes through its circumcenter; if there exists a point not on ℓ that satisfies the given equations, then the triangle must be equilateral.*

Set $x = PA, y = PB$, and $z = PC$, and assume the labels have been chosen so that $z \neq 0$. The given system of equations then becomes

$$a(x^2-z^2)+b(z^2-y^2)+c(y^2-x^2) = 0 \text{ and } a(z^2-y^2)+b(y^2-x^2)+c(x^2-z^2) = 0,$$

or, equivalently,

$$ax^2 - by^2 + cy^2 - cx^2 = (a-b)z^2 \quad (1)$$

$$-ay^2 + by^2 - bx^2 + cx^2 = (c-a)z^2 \quad (2)$$

Case 1. If $a \neq b, c$, then we can multiply the first equation by $c-a$, the second by $a-b$, and subtract to obtain

$$(c-a)(ax^2 - by^2 + cy^2 - cx^2) - (a-b)(-ay^2 + by^2 - bx^2 + cx^2) = 0,$$

or, after regrouping,

$$(x^2 - y^2)(bc + ca + ab - a^2 - b^2 - c^2) = 0.$$

Because $a \neq b, c$ the AM-GM inequality implies that $bc+ca+ab-a^2-b^2-c^2 \neq 0$. Hence, $x^2 = y^2$, and (from (1))

$$z^2 = \frac{ax^2 - by^2 + cy^2 - cx^2}{a-b} = \frac{x^2(a-b)}{a-b} = x^2.$$

Consequently, $PA = PB = PC$; that is, P is equidistant from the three vertices so that it lies on the line ℓ. Observe that when $PA = PB = PC$, each of the three expressions in the problem statement are equal to $PA^2(a+b+c)$.

Case 2. If $a = b$ then equations (1) and (2) become

$$(a-c)(x^2 - y^2) = 0 \text{ and } (c-a)(x^2 - z^2) = 0.$$

If in addition, $x = y = z$, then again, P lies on ℓ. Otherwise, $c = a = b$ and ΔABC is equilateral, as claimed.

Also solved by MARIAN DINCĂ, Bucharest, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; M. A. PRASAD, India; and the proposer.

The solution submitted by the proposer clearly indicates that he had intended to exclude the circumcentre as a possible position for P (as in the restatement of the problem). Among the other three submissions, two dealt with specialized interpretations, and one simply provided the counterexample.

YEAR END FINALE

We have have come to the end of another volume. It has been another interesting year here at the journal. We have done some structural changes, moving from 8 issues per year, back to 10 issues per year. At the same time, we have gone from printing every issue to bundling two issues together to try to make up some of the money that the journal has been losing. On top of that, our name has changed back ***Crux Mathematicorum***.

Also, ***Mathematical Mayhem*** and ***Skoliad*** will be separating from ***Crux Mathematicorum***. The solutions to most problems from these sections that have appeared in print in ***CRUX with MAYHEM*** have appeared in Volume 38. The last of the solutions will appear in the first two issues of Volume 39. The original plan was that ***Mathematical Mayhem*** (with ***Skoliad***) would carry on separately and be expanded. Unfortunately, there is neither the personnel, nor the funds to do so at this point in time. Hopefully ***Mayhem*** and ***Skoliad*** will not have to sit on the shelf too long before they are brought back.

We have introduced several new features that have received positive feedback. The new ***Contest Corner*** was introduced to fill in the gap left by ***Mayhem*** and ***Skoliad***. This volume saw just problem proposals, but your solutions have been coming in and they will start to appear next issue, just as the the solutions to the new ***Olympiad Corner*** problems have appeared this volume.

Three new columns were also introduced this volume. Longtime ***Crux*** contributor MICHEL BATAILLE started his column ***Focus On ...*** looking at advanced problem solving techniques. I must thank Michel for this great new column, it has received much praise. We have also introduced a related column ***The Problem Solver's Toolkit*** that focuses on more standard techniques. Also appearing is the ***Problem of the Month*** which features a nice problem and its solution.

It is our plan that, through Volume 39, each of the new columns will appear every second issue, with an article appearing every second issue as well. Through Volume 39 we are hoping to make up some time, and clear some of the backlog so that as each new issue appears, the date on the front of your journal should be getting closer to the date on your calendar!

Once the backlog of ***Crux*** issues has been cleared and we are up to date, we are looking to expand some of our features. That can mean more problems in the ***Contest Corner*** and ***Olympiad Corner***. We are also hoping that some day both ***The Problem Solver's Toolkit*** and ***Problem of the Month*** will run every issue. Until the point in time when the content is fixed, there will be some slight variations in the page count, issue to issue.

I would like to thank our readers for their patience in this time of change and delays. The kind words and encouragement that you have sent to me are greatly appreciated. I will continue to work hard to make ***Crux Mathematicorum*** the best it can be.

I am not alone putting the journal together, there are a number of people that I must thank. First and foremost, I must thank the editorial staff. I thank my associate editor JEFF HOOPER and BILL SANDS, the editor-at-large, for their detailed proofreading. They always pick up many things I have missed. I thank CHRIS FISHER for his work as problems editor and for his nine part ***Recurring Crux Configurations*** which wrapped up this issue. His dedication and expertise are greatly appreciated. I thank EDWARD WANG for his work as problems editor as well as his continued contribution of nice problems to the problems section. Our readers greatly benefit from his contributions. I want to thank ANNA KUCZYNSKAI who has settled into her role as problems editor and provides me with great edited solutions. I thank EDWARD BARBEAU for his work as problems editor. I have known Ed for a long time and it has been great having him on the team. I thank NICOLAE STRUNGARU for his role as editor to the ***Olympiad Corner***. The role involved a greater commitment of time on his part and his efforts are greatly appreciated by the readers and by me. I also thank CHRIS GRANDISON for his work as a problems editor. Chris will be leaving us at the end of this issue, thank you for your years of dedication to ***Crux***.

I want to thank ROBERT CRAIGEN and RICHARD GUY who continue to give some "unofficial" help. They are a great help to me proofreading and commenting on new features.

I thank JEAN-MARC TERRIER and ROLLAND GAUDET for providing French translations. The translations are always done quickly and occasionally there are comments on how to improve the English part as well! I must also welcome ANDRÉ LADOUCEUR who will help with the translations of an increasing number of problems.

I thank ROBERT DAWSON for for his work as the articles editor and for providing me with a great set of articles this year. I thank AMAR SODHI for keeping me in good supply of interesting book reviews. Amar will be leaving at the end of this issue, he will be missed. Thanks for all your work over the years Amar!

I thank LILY YEN and MOGENS LEMVIG HANSEN for their work as ***Skoliad*** Editors and the great job they do. The last ***Skoliad*** will appear next issue and Lily and Mogens will be leaving after that point. Thank you for your professionalism and the dedication you have had to the column through the years.

A special thank you goes out to my ***Mayhem*** assistant editor LYNN MILLER for her dedication and extra help behind the scenes. When the last of the ***Mayhem*** problems appears in issue 2 of the next Volume, Lynn will continue on with the journal, thank goodness!

I thank the staff at the CMS head office in Ottawa. In particular I thank JOHAN RUDNICK, DENISE CHARRON, and STEVE LA ROCQUE. Their behind the scenes support of me at ***Crux Mathematicorum*** is greatly appreciated. I thank TAMI EHRLICH and the staff at Thistle Printing for taking the files that I send them and turning them into the journal you have in your hands.

I must also thank the OTTAWA DISTRICT SCHOOL BOARD for partnering with the CMS to allow me to work as the Editor-in-Chief. In particular I would like to thank director JENNIFER ADAMS, superintendent PINO BUFFONE, Human Resources Officer JENNIFER BALDELLI and principal KEVIN GILMORE for their support and encouragement.

I also must thank my wife JULIE and my sons SAMUEL and SIMON for their continuing support. You guys always have my back, even though I am a little strange at times.

Finally, I sincerely thank all of the readers of ***CRUX***. The journal is really a function of the contributions we get from our readers. I look forward to another n years of your contributions, letters and email. Now I have to start working on issue 1 . . .

Shawn Godin

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Crux Mathematicorum with Mathematical Mayhem

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