THE OLYMPIAD CORNER

No. 299

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The solutions to the problems are due to the editor by 1 July 2013.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions’ section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier of the University of Montreal for translations of the problems.

OC61. 46 squares of a $9 \times 9$ grid are coloured red. Prove that we can find a $2 \times 2$ square on the grid which contains at least 3 red squares.

OC62. Let $A, B, C, D$ be four non-coplanar points in space. The segments $AB, BC, CD$ and $DA$ are tangent to the same sphere. Prove that their four points of tangency are coplanar.

OC63. Prove that there exists a perfect square so that the sum of its digits is 2011.

OC64. Find all integer solutions of the equation

$$n^3 = p^2 - p - 1$$

where $p$ is prime.

OC65. Let $ABC$ be a triangle. $F$ and $L$ are two points on the side $AC$ such that $AF = LC < AC/2$. If $AB^2 + BC^2 = AL^2 + LC^2$ find $\angle FBL$.

OC61. 46 carrés d’une grille de format $9 \times 9$ sont coloriés en rouge. Montrer qu’on peut trouver, dans cette grille, un carré de format $2 \times 2$ pouvant contenir au moins 3 carrés rouges.


OC63. Montrer qu’il existe un carré parfait tel que la somme de ses chiffres est 2011.
OC64. Trouver toutes les solutions de l’équation
\[ n^3 = p^2 - p - 1 \]
avec \( n \) entier et \( p \) premier.

OC65. Soit \( ABC \) un triangle. \( F \) et \( L \) sont deux points sur le côté \( AC \) tel que \( AF = LC < AC/2 \). Si \( AB^2 + BC^2 = AL^2 + LC^2 \) trouver \( \angle FBL \).

OLYMPIAD SOLUTIONS

OC1. Find all positive integers \( w, x, y \) and \( z \) which satisfy \( w! = x! + y! + z! \).
\( \text{(Originally question \# 1 from the 1983 Canadian Mathematical Olympiad.)} \)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Curtis.

By symmetry we may assume that \( x \leq y \leq z \leq w \).

If \( y > x \) then \((x + 1)! \) divides \( y!, z! \) and \( w! \), thus \((x + 1)! \) must divide \( x! \), a contradiction. Thus we get \( x = y \).

We now break the problem in two cases:

Case 1: \( y < z \). In this case \((y + 1)! \) divides \( z! \) and \( w! \) thus it must divide \( x! + y! = 2y! \). This implies that \((y + 1)! \) divides \( 2 \), and hence \( y = 1 \). Our equation becomes then

\[ w! = z! + 2. \]

From here we get \( w > z \), and since \( z! \) divides \( w! \) it must also divide \( 2 \). Thus, as \( z > y = 1 \), we get that \( z = 2 \), and hence \( w! = 4 \), which is not possible.

We get no solution in this case.

Case 2: \( y = z \). In this case our equation becomes

\[ w! = 3x!. \]

Then \( w > x \) and hence

\[ (x + 1)! | w! = 3x! \Rightarrow x + 1 | 3. \]

Since \( x \geq 1 \), we get \( x = 2 \) which yields the solution \( x = y = z = 2 \) and \( w = 3 \).

Hence the only solution is \((2, 2, 2, 3)\).

[\text{Ed.: It is clear that } x, y, z < w \text{ thus } x!, y!, z! < (w - 1)! \text{. Hence we get}
\[ w! = x! + y! + z! \leq 3(w - 1)! \Rightarrow w \leq 3 \]
\text{with equality if and only if } x = y = z = w - 1 \text{. Thus, either } x = y = z = 2 \text{ and } w = 3 \text{ or } w \leq 2, \text{ in which case } x! + y! + z! \geq 3 > 2! \geq w! \text{ which yields no new solutions.}]\]
OC2. Suppose that \( f \) is a real-valued function for which
\[
f(xy) + f(y - x) \geq f(y + x)
\]
for all real numbers \( x \) and \( y \).

(a) Give a nonconstant polynomial that satisfies the condition.

(b) Prove that \( f(x) \geq 0 \) for all real \( x \).

(Originally question \# 3 from the 2007 Canadian Mathematical Olympiad.)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Titu Zvonaru, Comănești, Romania. We give the solution of Curtis.

(a) Let \( P(x) = x^2 + 4 \). Then
\[
P(xy) + P(y - x) - P(y + x)
= x^2y^2 + 4 + y^2 - 2xy + x^2 + 4 - x^2 - y^2 - 2xy - 4
= x^2y^2 - 4xy + 4 = (xy - 2)^2 \geq 0.
\]
Thus
\[
P(xy) + P(y - x) \geq P(y + x).
\]

(b) Let \( \alpha \) be any real number. Solving \( x + y = xy \) and \( y - x = \alpha \) yields
\[
x = \frac{2 + \sqrt{4 + \alpha^2} - \alpha}{2}; \quad y = \frac{2 + \sqrt{4 + \alpha^2} + \alpha}{2}.
\]
Setting these values in our inequality we get
\[
f(2 + \sqrt{\alpha^2 + 4}) + f(\alpha) \geq f(2 + \sqrt{\alpha^2 + 4}),
\]
hence \( f(\alpha) \geq 0 \) for any real number \( \alpha \).

OC3. Let \( ABCD \) be a convex quadrilateral with
\[
\angle CBD = 2\angle ADB, \\
\angle ABD = 2\angle CDB
\]
and \( AB = CB \).

Prove that \( AD = CD \).

(Originally question \# 4 from the 2000 Canadian Mathematical Olympiad.)

Solved by Michel Bataille, Rouen, France; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We give the solution of Geupel.
Let \( \alpha = \angle ADB, \beta = \angle CDB \), \( a = AB = CB \) and \( b = DB \). By the Law of Sines in the triangles \( ABD \) and \( CBD \) we get

\[
\frac{\sin(\alpha)}{\sin(\alpha + 2\beta)} = \frac{a}{b} = \frac{\sin(\beta)}{\sin(2\alpha + \beta)}.
\]

It follows that

\[
\sin(\alpha) \sin(2\alpha + \beta) = \sin(\beta) \sin(\alpha + 2\beta),
\]

and hence

\[
\cos(\alpha + \beta) - \cos(3\alpha + \beta) = \cos(\alpha + \beta) - \cos(\alpha + 3\beta).
\]

Thus

\[
\cos(3\alpha + \beta) - \cos(\alpha + 3\beta) = 0,
\]

or

\[
\sin(2\alpha + 2\beta) \sin(\alpha - \beta) = 0.
\]

Since \( ABCD \) is convex we have \( 0 < \angle ABC = 2\alpha + 2\beta < 180^\circ \) and thus \( \sin(2\alpha + 2\beta) \neq 0 \). This implies that \( \sin(\alpha - \beta) = 0 \), and hence, using \( 0 < \alpha + \beta < 90^\circ \), we get \( \alpha = \beta \). Then, the triangles \( ABC \) and \( CBD \) are congruent, which yields the desired equality.

[Ed.: The only place we use the fact that \( ABCD \) is convex is to deduce that \( \sin(2\alpha + 2\beta) \neq 0 \). If we drop the convexity requirement, the only other possibility is \( \angle ABC = 180^\circ \). In this case \( \triangle DAC \) is a right angle triangle, and \( B \) is the midpoint of \( AC \). Such a triangle is always a non-convex solution.]

**OC4.** Consider 70-digit numbers \( n \), with the property that each of the digits \( 1, 2, 3, \ldots, 7 \) appears in the decimal expansion of \( n \) ten times (and \( 8, 9 \) and \( 0 \) do not appear). Show that no number of this form can divide another number of this form.

(Originally question \# 1 from the 2011 Canadian Mathematical Olympiad.)

Solved by Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Oliver Geupel, Brühl, NRW, Germany; Konstantine Zelator, University of Pittsburgh, Pittsburgh, PA, USA; and Titu Zvonaru, Comănești, Romania. We provide the similar solutions of Curtis and Zvonaru.

For a positive integer \( n \), let \( s(n) \) denote the sum of the digits of \( n \). It is well known that \( n \equiv s(n) \pmod 9 \).

Suppose by contradiction that there are two numbers \( A > B \) as in the problem so that \( B \mid A \). Then, there exists a positive integer \( C \) so that \( A = BC \). Then, since \( A < 8 \cdot 10^{69} \) and \( B > 10^{69} \) we get \( 1 < C < 8 \).

We have \( s(A) = s(B) = 10 \cdot (1 + 2 + 3 + 4 + 5 + 6 + 7) = 280 \), and hence \( A \equiv B \equiv 1 \pmod 9 \). Thus \( 1 \equiv A \equiv BC \equiv C \pmod 9 \). But this contradicts \( 1 < C < 8 \). Since we got a contradiction, our assumption is wrong, and thus the claim of the problem is true.

OC5. Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0,$$
$$a_1 + a_2 \leq 100$$

and

$$a_3 + a_4 + \cdots + a_{100} \leq 100.$$

Determine the maximum possible value of $a_1^2 + a_2^2 + \cdots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ which achieve this maximum.

(Originally question # 5 from the 2000 Canadian Mathematical Olympiad.)

Solved by Michel Bataille, Rouen, France; and Oliver Geupel, Brühl, NRW, Germany. We give the solution of Bataille.

Let $Q = a_1^2 + a_2^2 + \cdots + a_{100}^2$. We show that the maximum value of $Q$ is 10,000, attained exactly when $(a_1, a_2, \ldots, a_{100}) = (100, 0, 0, \ldots, 0)$ or $(50, 50, 50, 50, 0, \ldots, 0)$. First, we calculate:

$$100a_2 - (a_3^2 + \cdots + a_{100}^2) \geq a_2(a_3 + \cdots + a_{100}) - (a_3^2 + \cdots + a_{100}^2)$$
$$= a_3(a_2 - a_3) + \cdots + a_{100}(a_2 - a_{100}) \geq 0.$$  

Thus,

$$100a_2 \geq a_3^2 + a_4^2 + \cdots + a_{100}^2. \quad (1)$$

On the other hand, since $0 \leq a_1 \leq 100 - a_2$, we have

$$a_1^2 + a_2^2 \leq (100 - a_2)^2 + a_2^2. \quad (2)$$

By addition, (1),(2) give

$$Q \leq (100 - a_2)^2 + a_2^2 + 100a_2 = 10,000 + 2a_2(a_2 - 50).$$

Now, the conditions $a_1 \geq a_2$ and $a_1 + a_2 \leq 100$ call for $a_2 \leq 50$, hence $2a_2(a_2 - 50) \leq 0$. It follows that $Q \leq 10,000$ and, if equality holds, then $a_2 = 0$ or $a_2 = 50$. In the former case, we have $a_3 = a_4 = \cdots = a_{100} = 0$ and since we must also have $a_1 = 100 - a_2$, we obtain $(a_1, a_2, \ldots, a_{100}) = (100, 0, 0, \ldots, 0)$. In the latter case, $a_1 = 100 - a_2 = 50$ and since equality holds in (1), $a_3 + \cdots + a_{100} = 100$ and for each $k \in \{3, 4, \ldots, 100\}$, $a_k = 0$ or $a_k = a_2 = 50$. Recalling that $a_3 \geq a_4 \cdots \geq a_{100}$, this implies that $a_3 = a_4 = 50$ and $a_k = 0$ for $k \geq 5$. Thus, $(a_1, a_2, \ldots, a_{100}) = (50, 50, 50, 50, 0, \ldots, 0)$. Conversely, the two 100-tuples $(100, 0, 0, \ldots, 0)$ and $(50, 50, 50, 50, 0, \ldots, 0)$ satisfy the conditions and give $Q = 10,000$.

Several solvers noted that some of the problems have appeared in past issues of the journal. As this first problem set came at a transition period for the Olympiad Corner, the Editor-in-Chief put the problem set together using CMO problems from the not so recent past (in most cases). Future problem sets will consist of problems that haven’t appeared in the journal before (we hope!).