THE CONTEST CORNER

No. 1

Shawn Godin

The Contest Corner is a new feature of Crux Mathematicorum. It will be filling the gap left by the movement of Mathematical Mayhem and Skoliad to a new on-line journal in 2013. The column can be thought of as a hybrid of Skoliad, The Olympiad Corner and the old Academy Corner from several years back. The problems featured will be from high school and undergraduate mathematics contests with readers invited to submit solutions. Readers’ solutions will begin to appear in the next volume.

Solutions can be sent to:

Shawn Godin
Cairine Wilson S.S.
975 Orleans Blvd.
Orleans, ON, CANADA
K1C 2Z5

or by email to

crux-contest@cms.math.ca.

The solutions to the problems are due to the editor by 1 July 2013.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, 7, and 9, English will precede French, and in issues 2, 4, 6, 8, and 10, French will precede English. In the solutions’ section, the problem will be stated in the language of the primary featured solution.

The editor thanks Rolland Gaudet of Université de Saint-Boniface, Winnipeg, MB for translating the problems from English into French.

CC1. A circle has centre $O$, diameter $AC$, and radius 1. A chord is drawn from $A$ to an arbitrary point $B$ (different from $A$) on the circle and extended to the point $P$ with $BP = 1$. Thus $P$ can take many positions. Let $S$ be the set of points $P$. Determine whether or not there is a circle on which all points of $S$ lie.

CC2. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Suppose that $f$ is continuous and that $\int_0^1 f(a+tu)dt = 0$ for every point $a \in \mathbb{R}^2$ and every vector $u \in \mathbb{R}^2$ with $\|u\| = 1$. Show that $f$ is constant.

CC3. All three sides of a right triangle are integers. Prove that the area of the triangle: is also an integer; is divisible by 3; and is even.
CC4. Suppose that \( n \geq 3 \). A sequence \( a_1, a_2, a_3, \ldots, a_n \) of \( n \) integers, the first \( m \) of which are equal to \(-1\) and the remaining \( p = n - m \) of which are equal to \( 1 \), is called an \( MP \) sequence. Consider all of the products \( a_ia_ja_k \) (with \( i < j < k \)) that can be calculated using the terms from an \( MP \) sequence \( a_1, a_2, a_3, \ldots, a_n \). Determine the number of pairs \( (m, p) \) with \( 1 \leq m \leq p \leq 1000 \) and \( m + p \geq 3 \) for which exactly half of these products are equal to \( 1 \).

CC5. Let \( ABCD \) be a parallelogram. We draw in the diagonal \( AC \). A circle is drawn inside \( \triangle ABC \) tangent to all three sides and touches side \( AC \) at a point \( P \). Draw in the line \( DP \). A circle of radius \( r_1 \) is drawn inside \( \triangle DAP \) tangent to all three sides and a circle of radius \( r_2 \) is drawn inside \( \triangle DCP \) tangent to all three sides. Prove that \[
\frac{r_1}{r_2} = \frac{AP}{PC}.
\]