MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a Mathematical Journal for and by High School and University Students. It continues, with the same emphasis, as an integral part of Crux Mathematicorum with Mathematical Mayhem.

The interim Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School, Orleans, ON). The Assistant Mayhem Editor is Lynn Miller (Cairine Wilson Secondary School, Orleans, ON). The other staff members are Ann Arden (Osgoode Township District High School, Osgoode, ON), Nicole Diotte (Windsor, ON), Monika Khbeis (Our Lady of Mt. Carmel Secondary School, Mississauga, ON) and Daphne Shani (Bell High School, Nepean, ON).

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Mayhem Year End Wrap Up

Shawn Godin

Hello Mayhem readers. Another “year” has ended and it marks the end of another chapter in the story of Mathematical Mayhem. Mayhem was created “by students, for students” in the fall of 1988 and was published 5 times a year for 8 years. After losing its funding, Mayhem joined Crux in 1997, volume 23. Now, with times changing, it is time for the journal to make yet another change.

Mayhem will revert to a stand alone journal that appears 5 times a year and follows the school year (issues in September, November, January, March and May). The difference is that Mayhem will exist on-line. The plan is to expand the journal beyond just problems to include columns, articles, interactive material and possibly even videos. We are working hard toward the relaunch of Mayhem which should occur some time in early 2013. Keep your eye on the CMS web site and the Crux Facebook page for updates.

At this point I need to thank the Mayhem staff for their help preparing the material for each issue. The problems editors ANN ARDEN, NICOLE DIOTTE, MONIKA KHBEIS and DAPHNE SHANI who sift through all the submitted solutions and prepare the featured solutions, your work is very much appreciated. Also to my assistant editor LYNN MILLER, for your work behind the scenes and preparing solutions, I want to thank you from the bottom of my heart. You are always there when I need a little extra help.

I wish all the best to our readers. I look forward to receiving your problem proposals and solutions. Keep your eye open for the new Mayhem.

Shawn Godin
Mayhem Problems

Please send your solutions to the problems in this edition by 15 November 2012. Solutions received after this date will only be considered if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Rolland Gaudet, Université de Saint-Boniface, Winnipeg, MB, for translating the problems from English into French.

M513. Proposé par l’Équipe de Mayhem.

Un grillage triangulaire équilatéral consiste de chevilles espacées d’un centimètre l’une de l’autre, tel qu’indiqué au schéma. Des bandes élastiques sont placées autour des chevilles de façon à former des triangles équilatéraux; deux tels triangles équilatéraux à deux centimètres de côté sont illustrés au schéma. Combien de triangles équilatéraux différents sont possibles?

M514. Proposé par Dragoljub Milošević, Gornji Milanovac, Serbie.

Le nonagone \( ABCDEFGHI \) est régulier. Démontrer que \( AE - AC = AB \).

M515. Proposé par Titu Zvonaru, Comănești, Roumanie.

Sans utiliser les techniques du calcul différentiel, déterminer les valeurs minimales et maximales de

\[
\frac{2x}{x^2 + 2x + 2}
\]

où \( x \) est un nombre réel.

M516. Proposé par Syd Bulman-Fleming et Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Démontrer que pour tout \( k \) entier non nul il existe au moins quatre paires ordonnées d’entiers, \((x, y)\), telles que

\[
\frac{y^2 - 1}{x^2 - 1} = k^2 - 1.
\]

M517. Proposé par Šefket Arslanagić, Université de Sarajevo, Sarajevo, Bosnie et Herzégovine.

Déterminer toutes les solutions réelles de l’équation

\[
3\sqrt{x + y} + 2\sqrt{8 - x} + \sqrt{6 - y} = 14.
\]
M518. Sélectionné à partir de concours mathématiques.

Un nombre de carrés unitaires sont placés sur une ligne, tel qu’indiqué au schéma ci-bas.


M513. Proposed by the Mayhem Staff.

An equilateral triangular grid is formed by removable pegs that are one centimetre apart as shown in the diagram. Elastic bands may be attached to pegs to form equilateral triangles, two different equilateral triangles two centimetres on each side are shown in the diagram. How many different equilateral triangles are possible?

M514. Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.

Nonagon $ABCDEFGHI$ is regular. Prove that $AE - AC = AB$.

M515. Proposed by Titu Zvonaru, Comănești, Romania.

Without using calculus, determine the minimum and maximum values of

$$\frac{2x}{x^2 + 2x + 2}$$

where $x$ is a real number.

M516. Proposed by Syd Bulman-Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

Show that for any given nonzero integer $k$ there exists at least four distinct ordered pairs $(x, y)$ of integers such that

$$\frac{y^2 - 1}{x^2 - 1} = k^2 - 1.$$
**M517.** Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Find all real solutions of the equation

\[ 3\sqrt{x + y} + 2\sqrt{8 - x} + \sqrt{6 - y} = 14. \]

**M518.** Selected from a mathematics competition.

A number of unit squares are placed in a line as shown in the diagram below.

Let \( O \) be the bottom left corner of the first square and let \( P \) and \( Q \) be the top right corners of the 2011th and 2012th squares respectively. When \( P \) and \( Q \) are connected to \( O \) they intersect the right side of the first square at \( X \) and \( Y \) respectively. Determine the area of triangle \( OXY \).

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**Mayhem Solutions**

**M476.** Proposed by the Mayhem Staff

Define \( s(n) \) to be the sum of the digits of the positive integer \( n \). For example, \( s(2011) = 2 + 0 + 1 + 1 = 4 \). Determine the number of four-digit positive integers \( n \) with \( s(n) = 4 \).

*Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.*

There are five ways to write 4 as the sum of positive integers; namely 4, \( 3 + 1 \), \( 2 + 2 \), \( 2 + 1 + 1 \), and \( 1 + 1 + 1 + 1 \). If \( s(n) = 4 + 0 + 0 + 0 \), then \( n = 4000 \) is the only such integer. If \( s(n) = 3 + 1 \), then there are six possible values for \( n \); namely \( n = 3100, 3010, 3001, 1300, 1030, \) or \( 1003 \). If \( s(n) = 2 + 2 \), then the three possible values for \( n \) are \( n = 2200, 2020, \) or \( 2002 \). If \( s(n) = 2 + 1 + 1 \), then the nine possible values for \( n \) are \( n = 2101, 2110, 1021, 1012, 1102, 1120, 1210, \) or \( 1201 \). Finally, if \( s(n) = 1 + 1 + 1 + 1 \), then \( n = 1111 \) is the only such integer. Hence, there are 20 positive integers with \( s(n) = 4 \).

*Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ALICIA GÓMEZ GÓMEZ, Club Mathématique de l’Instituto de Ecuación Secundaria No. 1, Requena-Valencia,*
Let \( m \) be an integer parameter such that the equation \( x^2 - mx + m + 8 = 0 \) has one integer root. Determine the value of the parameter \( m \).

**Solution by George Apostolopoulos, Messolonghi, Greece.**

If there is only one root then the discriminant must be zero, so

\[
(-m)^2 - 4(m + 8) = 0 \Rightarrow (m - 2)^2 = 36
\]

\[
\Rightarrow m - 2 = \pm 6
\]

\[
\Rightarrow m = 8 \text{ or } m = -4.
\]

Also solved by DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA. Two incorrect solutions were submitted.

**M478. Proposed by the Mayhem Staff**

Consider the set of points \((x, y)\) in the plane such that

\[ x^2 + y^2 - 22x - 4y + 100 = 0. \]

Let \( P \) be the point in this set for which \( \frac{y}{x} \) is the largest. Determine the distance of \( P \) from the origin.

**Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.**

We recognize the given set of points as the circle with centre at \((11, 2)\) and radius 5 and whose parametric equations are

\[
x = 11 + 5 \cos \theta, \quad y = 2 + 5 \sin \theta
\]

Denote \( \tan \frac{\theta}{2} \) by \( t \). Since \( \sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \) and \( \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \), equations (1) are equivalent to

\[
x = \frac{6t^2 + 16}{1 + t^2}, \quad y = \frac{2t^2 + 10t + 2}{1 + t^2}
\]

Hence, \( \frac{y}{x} = \frac{t^2 + 5t + 1}{3t^2 + 8} \), the derivative \( \frac{dy}{dt} \) is \( \frac{5(-3t^2 + 2t + 8)}{(3t^2 + 8)^2} \), and the critical values are solutions of \(-3t^2 + 2t + 8 = 0\) or \( t = -\frac{4}{3} \) or \( 2 \). The value \( t = -\frac{4}{3} \) corresponds to a minimum, and \( t = 2 \) corresponds to a maximum. We find \( x \) and \( y \) for \( t = 2 \) to be \( x = 8 \), \( y = 6 \), so that the distance of \( P \) from the origin is

\[
\sqrt{x^2 + y^2} = \sqrt{8^2 + 6^2} = 10
\]
Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARD PEIRÓ, IES “Abastos”, Valencia, Spain; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. Three incorrect solutions were submitted.

M479. Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania

Let \( A = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 2011 = 2011! \).

(a) Determine the largest positive integer \( n \) for which \( 3^n \) divides exactly into \( A \).

(b) Determine the number of zeroes at the end of the base 10 representation of \( A \).

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

(a) The exponent \( n \) of the highest power of 3 that divides exactly into \( 2011! \) is given by \( n = \sum_{k=1}^{\infty} \left\lfloor \frac{2011}{3^k} \right\rfloor \). One calculates \( \left\lfloor \frac{2011}{3} \right\rfloor = 670 \), \( \left\lfloor \frac{2011}{9} \right\rfloor = 223 \), \( \left\lfloor \frac{2011}{27} \right\rfloor = 74 \), \( \left\lfloor \frac{2011}{81} \right\rfloor = 24 \), \( \left\lfloor \frac{2011}{243} \right\rfloor = 8 \), \( \left\lfloor \frac{2011}{729} \right\rfloor = 2 \), and if \( k \geq 7 \), then \( \left\lfloor \frac{2011}{3^k} \right\rfloor = 0 \). Therefore, \( n = 670 + 223 + 74 + 24 + 8 + 2 = 1001 \).

(b) The number of zeroes with which the decimal representation of \( 2011! \) terminates is equal to the exponent, \( m \), of the highest power of 10 that divides \( 2011! \). Furthermore, \( m \) is also the exponent of the highest power of 5 that divides \( 2011! \), that is, \( m = \sum_{k=1}^{\infty} \left\lfloor \frac{2011}{5^k} \right\rfloor = 402 + 80 + 16 + 3 = 501 \).

Hence, the base 10 representation of \( 2011! \) ends in 501 zeroes.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICARD PEIRÓ, IES “Abastos”, Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.

M480. Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Let \( x, y, \) and \( k \) be positive numbers such that \( x^2 + y^2 = k \). Determine the minimum possible value of \( x^6 + y^6 \) in terms of \( k \).

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

Note that \( x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4) = k((x^2 + y^2)^2 - 3x^2y^2) = k(k^2 - 3x^2y^2) \) attains its minimum possible value if and only if \( -3x^2y^2 \) attains its minimum value, or equivalently, if and only if \( x^2y^2 \) attains its maximum value. By the arithmetic mean-geometric mean inequality, \( x^2y^2 \leq \left( \frac{x^2 + y^2}{2} \right)^2 = \frac{k^2}{4} \) with equality if and only if \( x = y \), that is \( x^2y^2 \) attains its maximum value if and only if \( x = y = \sqrt{\frac{k}{2}} \), so the minimum possible value of \( x^6 + y^6 \) is \( x^6 + y^6 = k \left( k^2 - 3 \frac{k^2}{4} \right) = \frac{k^3}{4} \).
Suppose that \(a\), \(b\), and \(x\) are real numbers with \(ab \neq 0\) and \(a + b \neq 0\). If 
\[
\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a + b},
\]
determine the value of \(\frac{\sin^6 x}{a^3} + \frac{\cos^6 x}{b^3}\) in terms of \(a\) and \(b\).

Solution by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

From 
\[
\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a + b},
\]
it follows
\[
\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} - \frac{1}{a + b} = 0
\]
\[
\Rightarrow b(a + b) \sin^4 x + a(a + b) \cos^4 x - ab = 0
\]
\[
\Rightarrow b^2 \sin^4 x + a^2 \cos^4 x + ab(\sin^4 x + \cos^4 x - 1) = 0
\]
\[
\Rightarrow b^2 \sin^4 x + a^2 \cos^4 x + ab(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x - 2 \sin^2 x \cos^2 x - 1 = 0
\]
\[
\Rightarrow b^2 \sin^4 x + a^2 \cos^4 x + ab(1 - 2 \sin^2 x \cos^2 x - 1) = 0
\]
\[
\Rightarrow b^2 \sin^4 x + a^2 \cos^4 x - 2ab \sin^2 x \cos^2 x = 0
\]
\[
\Rightarrow (b \sin^2 x - a \cos^2 x)^2 = 0.
\]

Therefore we have
\[
\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a + b} \Rightarrow (b \sin^2 x - a \cos^2 x)^2 = 0,
\]
and consequently, 
\[
\frac{\sin^2 x}{a} = \frac{\cos^2 x}{b} = \frac{1}{a + b}.
\]
Finally we obtain that
\[
\frac{\sin^6 x}{a^3} + \frac{\cos^6 x}{b^3} = \left(\frac{1}{a + b}\right)^3 + \left(\frac{1}{a + b}\right)^3
\]
\[
= \frac{2}{(a + b)^3}.
\]