M517. Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Find all real solutions of the equation

\[ 3\sqrt{x + y + 2\sqrt{8 - x + \sqrt{6 - y}}} = 14. \]

M518. Selected from a mathematics competition.

A number of unit squares are placed in a line as shown in the diagram below.

Let \( O \) be the bottom left corner of the first square and let \( P \) and \( Q \) be the top right corners of the 2011th and 2012th squares respectively. When \( P \) and \( Q \) are connected to \( O \) they intersect the right side of the first square at \( X \) and \( Y \) respectively. Determine the area of triangle \( OXY \).

---

Mayhem Solutions

M476. Proposed by the Mayhem Staff

Define \( s(n) \) to be the sum of the digits of the positive integer \( n \). For example, \( s(2011) = 2 + 0 + 1 + 1 = 4 \). Determine the number of four-digit positive integers \( n \) with \( s(n) = 4 \).

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

There are five ways to write 4 as the sum of positive integers; namely 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1. If \( s(n) = 4 \), then there are six possible values for \( n \); namely \( n = 3100, 3010, 3001, 1300, 1030, \) or 1003. If \( s(n) = 2 + 2 \), then the three possible values for \( n \) are \( n = 2200, 2020, \) or 2002. If \( s(n) = 2 + 1 + 1 \), then the nine possible values for \( n \) are \( n = 2111, 2101, 2110, 1021, 1012, 1102, 1120, 1210, \) or 1201. Finally, if \( s(n) = 1 + 1 + 1 + 1 \), then \( n = 1111 \) is the only such integer. Hence, there are 20 positive integers with \( s(n) = 4 \).

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; ALICIA GÓMEZ GÓMEZ, Club Mathématique de l’Instituto de Ecuación Secundaria No. 1, Requena-Valencia,
Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania

Let \( m \) be an integer parameter such that the equation \( x^2 - mx + m + 8 = 0 \) has one integer root. Determine the value of the parameter \( m \).

Solution by George Apostolopoulos, Messolonghi, Greece.

If there is only one root then the discriminant must be zero, so

\[
(-m)^2 - 4(m + 8) = 0 \iff (m - 2)^2 = 36 \iff m - 2 = \pm 6 \iff m = 8 \text{ or } m = -4.
\]

Also solved by DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA. Two incorrect solutions were submitted.

Proposed by the Mayhem Staff

Consider the set of points \((x, y)\) in the plane such that

\[
x^2 + y^2 - 22x - 4y + 100 = 0.
\]

Let \( P \) be the point in this set for which \( \frac{y}{x} \) is the largest. Determine the distance of \( P \) from the origin.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

We recognize the given set of points as the circle with centre at \((11, 2)\) and radius 5 and whose parametric equations are

\[
x = 11 + 5 \cos \theta, \quad y = 2 + 5 \sin \theta
\]

Denote \( \tan \frac{\theta}{2} \) by \( t \). Since \( \sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \) and \( \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \), equations (1) are equivalent to

\[
x = \frac{6t^2 + 16}{1 + t^2}, \quad y = \frac{2t^2 + 10t + 2}{1 + t^2}
\]

Hence, \( \frac{y}{x} = \frac{t^2 + 5t + 1}{3t^2 + 8} \), the derivative \( \frac{d}{dt} \left( \frac{y}{x} \right) \) is \( \frac{5(-3t^2 + 2t + 8)}{(3t^2 + 8)^2} \), and the critical values are solutions of \(-3t^2 + 2t + 8 = 0\) or \( t = -\frac{4}{3}, 2 \). The value \( t = -\frac{4}{3} \) corresponds to a minimum, and \( t = 2 \) corresponds to a maximum. We find \( x \) and \( y \) for \( t = 2 \) to be \( x = 8, \ y = 6 \), so that the distance of \( P \) from the origin is

\[
\sqrt{x^2 + y^2} = \sqrt{8^2 + 6^2} = 10
\]
Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; RICARD PEIRÓ, IES “Abastos”, Valencia, Spain; and PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy. Three incorrect solutions were submitted.

M479. Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania

Let \( A = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 2011 = 2011! \).

(a) Determine the largest positive integer \( n \) for which \( 3^n \) divides exactly into \( A \).

(b) Determine the number of zeroes at the end of the base 10 representation of \( A \).

Solution by David E. Manes, SUNY at Oneonta, Oneonta, NY, USA.

(a) The exponent \( n \) of the highest power of 3 that divides exactly into \( 2011! \) is given by 
\[ n = \sum_{k=1}^{\infty} \left\lfloor \frac{2011}{3^k} \right\rfloor. \]
One calculates 
\[ \left\lfloor \frac{2011}{3} \right\rfloor = 670, \quad \left\lfloor \frac{2011}{9} \right\rfloor = 223, \quad \left\lfloor \frac{2011}{27} \right\rfloor = 74, \quad \left\lfloor \frac{2011}{81} \right\rfloor = 24, \quad \left\lfloor \frac{2011}{243} \right\rfloor = 8, \quad \left\lfloor \frac{2011}{729} \right\rfloor = 2, \]
and if \( k \geq 7 \), then 
\[ \left\lfloor \frac{2011}{3^k} \right\rfloor = 0. \]
Therefore, 
\[ n = 670 + 223 + 74 + 24 + 8 + 2 = 1001. \]

(b) The number of zeroes with which the decimal representation of \( 2011! \) terminates is equal to the exponent, \( m \), of the highest power of 10 that divides \( 2011! \). Furthermore, \( m \) is also the exponent of the highest power of 5 that divides \( 2011! \), that is, 
\[ m = \sum_{k=1}^{\infty} \left\lfloor \frac{2011}{5^k} \right\rfloor = 402 + 80 + 16 + 3 = 501. \]
Hence, the base 10 representation of \( 2011! \) ends in 501 zeroes.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICARD PEIRÓ, IES “Abastos”, Valencia, Spain; and BRUNO SALGUEIRO FANEGO, Viveiro, Spain.

M480. Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Let \( x, y \), and \( k \) be positive numbers such that \( x^2 + y^2 = k \). Determine the minimum possible value of \( x^6 + y^6 \) in terms of \( k \).

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

Note that 
\[ x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4) = k((x^2 + y^2)^2 - 3x^2y^2) = k(k^2 - 3x^2y^2) \]
attains its minimum possible value if and only if \( -3x^2y^2 \) attains its minimum value, or equivalently, if and only if \( x^2y^2 \) attains its maximum value. By the arithmetic mean-geometric mean inequality, 
\[ x^2y^2 \leq \left( \frac{x^2 + y^2}{2} \right)^2 = \frac{k^2}{4} \]
with equality if and only if \( x = y \), that is \( x^2y^2 \) attains its maximum value if and only if \( x = y = \sqrt{\frac{k}{2}} \), so the minimum possible value of \( x^6 + y^6 \) is 
\[ x^6 + y^6 = k \left( k^2 - 3 \frac{k^2}{4} \right) = \frac{k^3}{4}. \]
Suppose that \( a, b, \) and \( x \) are real numbers with \( ab \neq 0 \) and \( a + b \neq 0 \). If \( \frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a + b} \), determine the value of \( \frac{\sin^6 x}{a^3} + \frac{\cos^6 x}{b^3} \) in terms of \( a \) and \( b \).

Solution by Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

From \( \frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a + b} \), it follows

\[
\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} - \frac{1}{a + b} = 0
\]

\[
\Rightarrow \quad b(a + b) \sin^4 x + a(a + b) \cos^4 x - ab = 0
\]

\[
\Rightarrow \quad b^2 \sin^4 x + a^2 \cos^4 x + ab(\sin^4 x + \cos^4 x - 1) = 0
\]

\[
\Rightarrow \quad b^2 \sin^4 x + a^2 \cos^4 x + ab(\sin^4 x + \cos^4 x - 1) = 0
\]

\[
\Rightarrow \quad b^2 \sin^4 x + a^2 \cos^4 x + ab(\sin^2 x + \cos^2 x) = 0
\]

\[
\Rightarrow \quad (b \sin^2 x - a \cos^2 x)^2 = 0.
\]

Therefore we have

\[
\frac{\sin^4 x}{a} + \frac{\cos^4 x}{b} = \frac{1}{a + b} \Rightarrow (b \sin^2 x - a \cos^2 x)^2 = 0,
\]

and consequently, \( \frac{\sin^2 x}{a} = \frac{\cos^2 x}{b} = \frac{1}{a + b} \). Finally we obtain that

\[
\frac{\sin^6 x}{a^3} + \frac{\cos^6 x}{b^3} = \left( \frac{1}{a + b} \right)^3 + \left( \frac{1}{a + b} \right)^3 = \frac{2}{(a + b)^3}.
\]