SKOLIAD  No. 137

Lily Yen and Mogens Hansen

Please send your solutions to problems in this Skoliad by December 15, 2012. A copy of CRUX with Mayhem will be sent to one pre-university reader who sends in solutions before the deadline. The decision of the editors is final.

Our contest this month is the Mathematics Association of Quebec Contest, Secondary level, 2011. Our thanks go to Denis Lavigne, Royal Military College Saint-Jean, Quebec, for providing us with this contest and for permission to publish it.

Concours de l’Association mathématique du Québec, 2011
Ordre secondaire
Durée : 3 heures

1. Otto aime tellement les palindromes (les nombres qui demeurent les mêmes lorsqu’on inverse l’ordre de leurs chiffres) qu’il a concocté l’alphamétique suivant :

   AMQMA \times 6 = LUCIE.

Trouver les valeurs des huit chiffres.

(N.B. Un alphamétique est un petit casse-tête mathématique qui consiste en une équation où les chiffres sont remplacés par des lettres. Le résoudre consiste à trouver quelle lettre correspond à quel chiffre pour que l’équation soit vraie. Dans le problème, le même chiffre ne peut être représenté par deux lettres différentes et une lettre représente toujours le même chiffre. Bien entendu, un nombre ne doit jamais commencer par zéro. Par exemple, l’alphamétique PAPA + PAPA = MAMAN a pour solution \( P = 7, A = 5, M = 1 \) et \( N = 0 \). Ainsi, en remplaçant les lettres par les chiffres, on a bien \( 7575 + 7575 = 15150 \).)

2. Anik roule toujours à 108 km/h sur l’autoroute. En se rendant à un concours de mathématique, elle dépasse un train qui longe l’autoroute et qui se dirige dans le même sens qu’elle. Elle remarque qu’elle met exactement 77 secondes à le dépasser, i.e. franchir la distance qui sépare la queue du train et sa tête. Arrivée à destination, elle réalise qu’elle a oublié sa calculatrice, alors elle rebrousse chemin. Elle recroise alors le train, qui roule à la même vitesse, et cette fois, elle met exactement sept secondes à parcourir le train de la tête à la queue. Quelle est la longueur du train ?

3. À un coin de rue, le feu de circulation reste vert pendant 30 secondes et rouge pendant 30 secondes (on suppose le temps du feu jaune inclus à même le temps du feu vert). Combien de temps perd-on, en moyenne, à attendre à ce coin de rue ? Justifier.

4. Combien y a-t-il de nombres entiers entre 0 et 999 (inclusivement) dont l’écriture décimale ne contient aucun 7 ? Quelle est la somme de ces nombres ?
5. Dans un cercle de rayon $r$, deux cordes $AB$ et $CD$ se coupent perpendiculairement en $X$. Montrer que 

$$ |XA|^2 + |XB|^2 + |XC|^2 + |XD|^2 = 4r^2. $$

6. $173^3 = 5177717$, $192^3 = 7077888$ et $1309^3 = 2242946629$ sont trois exemples d’entiers $N$ dont le cube compte le même nombre de chiffres différents que $N$ lui-même. Mais existe-t-il des entiers qui contiennent plus de chiffres différents que leur cube ? Oui : le nombre $13798$ compte cinq chiffres différents tandis que son cube, $2626929525592$ n’en compte que quatre ($2$ ; $5$ ; $6$ et $9$). On dira d’un tel nombre (quand il contient plus de chiffres différents que son cube) qu’il est déficient. Montrer qu’il y a une infinité d’entiers déficience.

7. Sachant que le système d’équations 

$$ x = \sqrt{11 - 2yz}, \quad y = \sqrt{12 - 2xz}, \quad z = \sqrt{13 - 2xy} $$

possède des solutions réelles, que vaut $x + y + z$ ?

**Mathematics Association of Quebec Contest, 2011**

**Secondary level**

3 hours allowed

1. Otto likes palindromes (numbers that read the same forwards and backwards) so much that he has constructed this alphametic:

$$ AMQMA \times 6 = LUCIE. $$

Find the values of the eight digits.

(Recall that an alphametic is a small mathematical puzzle consisting of an equation in which the digits have been replaced by letters. The task is to identify the value of each letter in such a way that the equation comes out true. Different letters have different values, different digits are represented by different letters, and no number begins with a zero. For example, the alphametic $PAPA + PAPA = MAMAN$ has the solution $P = 7$, $A = 5$, $M = 1$, and $N = 0$, yielding $7575 + 7575 = 15150$.)

2. Anik is going $108$ km/h on the highway. On her way to a math contest, she passes a train that travels beside the highway in the same direction as Anik. She notices that it takes her exactly $77$ seconds to pass the train from the rear to the front. Upon arrival, she finds that she has forgotten her calculator and turns back. She again passes the train, which still travels at the same speed. This time it takes her seven seconds to pass from the front of the train to the rear. How long is the train?
3. At an intersection, the traffic light is red for 30 seconds and green for 30 seconds. (Ignore the yellow light.) How long do you have to wait, on average, at the intersection? Justify your answer.

4. How many integers from 0 to 999 (inclusive) do not contain the digit 7? What is the sum of these numbers?

5. In a circle with radius \( r \), the two chords \( AB \) and \( CD \) intersect at a right angle at \( X \). Show that
\[
|XA|^2 + |XB|^2 + |XC|^2 + |XD|^2 = 4r^2.
\]

6. The cubes \( 173^3 = 517717 \), \( 192^3 = 7077888 \) and \( 1309^3 = 2242946629 \) are examples of a whole number \( N \) that contains as many different digits as its cube, \( N^3 \). If \( N^3 \) contains fewer different digits than \( N \), then \( N \) is said to be deficient. For example, 13798 has five different digits, while its cube, 2626929525592, has four (2, 5, 6, and 9), so 13798 is deficient. Show that there are infinitely many deficient whole numbers.

7. If \( x, y, \) and \( z \) are real numbers such that
\[
x = \sqrt{11 - 2yz}, \quad y = \sqrt{12 - 2xz}, \quad \text{and} \quad z = \sqrt{13 - 2xy}
\]
what is the value of \( x + y + z \)?


1. Sonja has nine cards on which the nine smallest two-digit prime numbers are printed. She wants to order these cards in such a way that neighbouring cards always differ by a power of 2. In how many ways can Sonja order her cards?

\textit{Solution by Elisa Kuan, student, Meadowridge School, Maple Ridge, BC.}

The first nine two-digit primes are 11, 13, 17, 19, 23, 29, 31, 37, and 41. The relevant powers of 2 are 1, 2, 4, 8, 16, and 32. In the figure, those primes that differ by a power of 2 are connected. Since 41 is only connected to one other prime, any arrangement of these primes must begin (or end) 41, 37, 29.

If the arrangement then continues with 31, it must go on as 41, 37, 29, 31, 23, 19. At 19 you again have a choice: 41, 37, 29, 31, 23, 19, 17, 13, 11 or 41, 37, 29, 31, 23, 19, 11, 13, 17. Both of these arrangements work out (that is, they use all nine primes).

If the arrangement instead has 13 following 29, you immediately have the choice: 11 or 17. In either case, you must go on to 19, so now you have 41, 37, 29, 13, 11, 19 or 41, 37, 29, 13, 17, 19. If you here go to 17 or 11 (whichever is
available), then 23 and 31 become stranded. If, on the other hand, you continue as 41, 37, 29, 13, 11/17, 19, 23, 31, then 11 or 17 becomes stranded.

Thus, the only possible arrangements are:

\[ 41, 37, 29, 31, 23, 19, 17, 13, 11; \]
\[ 41, 37, 29, 31, 23, 19, 11, 13, 17; \]

and these reversed, so Sonja can arrange her cards in four ways.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC; and SZERA PINTER, student, Moscrop Secondary School, Burnaby, BC.

2. A 50 cm by 30 cm by 28 cm box contains wooden blocks that all measure 10 cm by 9 cm by 7 cm. At most how many blocks can fit in the box? Explain how to fit that many blocks into the box.

Solution by Jay Chau, student, Burnaby Mountain Secondary School, Burnaby, BC.

The volume of the box is \( 50 \cdot 30 \cdot 28 = 42000 \) cm\(^3\), and the volume of each block is \( 10 \cdot 9 \cdot 7 = 630 \) cm\(^3\), so there is room for at most \( \left\lfloor \frac{42000}{630} \right\rfloor = \left\lfloor \frac{200}{3} \right\rfloor = 66 \frac{2}{3} \) blocks.

The figure shows that fitting 66 blocks is indeed possible.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

You may well wonder, how our solver found a way to squeeze all 66 blocks into the box. As a first try, you might line the 10-side of the blocks against the 50-side of the box and the 7-side against the 28-side since both work out perfectly. However, this leaves room for only \( \left\lfloor \frac{200}{3} \right\rfloor = 3 \) layers, so only \( 5 \cdot 4 \cdot 3 = 60 \) blocks. The trouble is that the 9-side of the blocks does not fit perfectly against any side of the box. The 50-side has the most wiggle room, so fitting as many 9’s as convenient could provide a better fit. Trying out different numbers of 9’s, you will find that \( 4 \cdot 9 + 2 \cdot 7 = 50 \), which leads to our solver’s solution.
3. Five distinct positive numbers are given. Forming all possible sums of two of these numbers you obtain seven different sums. Show that the sum of the five original numbers is divisible by 5.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Suppose the five distinct numbers are $a$, $b$, $c$, $d$, and $e$, and that $a < b < c < d < e$. Then

\[ a + b < a + c < a + d < a + e < b + e < c + e < d + e. \]

This is already seven distinct sums, so the remaining three sums, $b + c$, $b + d$, and $c + d$ must be in the seven sums listed above.

Since $a + d < b + d < b + e$, it follows from the long inequality above that $b + d = a + e$. Since $a + e = b + d < c + d < c + e$, it similarly follows that $c + d = b + e$. Finally, since $a + c < b + c < b + d = a + e$, it follows that $b + c = a + d$. That is, $b + d = a + e$, $c + d = b + e$, and $b + c = a + d$.

If you add the second and third of these equations and subtract the first, you get that $(b + c + d + e) - (a + e) = (b + d) + (a + d) - (a + e)$, so $2c = b + d$. On the other hand, $a + e = b + d$, so $a + e = 2c$. Thus $a + b + c + d + e = (a + e) + (b + d) + c = 2c + 2c + c = 5c$, which clearly is divisible by 5.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

As our solver clearly assumes (as does every other submitted solution), the five numbers should be integers. The problem should have made this clear; we apologise for the omission.

4. Three squares are arranged as in the figure. Show that the two shaded triangles have the same area.

\[ \text{Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.} \]

Let $a$ and $b$ denote the side lengths of the two outer squares, label the vertices as in the figure, and extend the top side of the right-hand square until it meets the vertical line through $D$. 

\[ \text{Solution by Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.} \]

Let $a$ and $b$ denote the side lengths of the two outer squares, label the vertices as in the figure, and extend the top side of the right-hand square until it meets the vertical line through $D$. 

Since the sides of $\triangle ABE$ are parallel to the corresponding sides of $\triangle DCF$ and $|AB| = |DC|$, it follows that $\triangle ABE \cong \triangle DCF$. Hence the distance from $D$ to the base line is $a + b$.

If you now use the horizontal side of each shaded triangle as the base, then the left-hand shaded triangle has base $a$ and height $b$, while the right-hand shaded triangle has base $b$ and height $a$. Therefore the area of either shaded triangle is $\frac{1}{2}ab$.

Also solved by GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

5. Triangle $\triangle ABC$ is isosceles and $\angle ACB = 90^\circ$. The point $D$ is on the line $AC$ beyond $C$, and the point $E$ is on the line $CB$ beyond $B$. Show that $|CD| = |CE|$ if line $BD$ is perpendicular to line $AE$.

Solution by Rowena Ho, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

In the figure in the statement of the problem, $BD$ does not seem perpendicular to $AE$, so move $D$ farther out. Moreover, let $F$ denote the intersection between $BD$ and $AE$.

Now similar triangles do the work: $\angle EBF = \angle DBC$ and $\angle BFE = 90^\circ = \angle BCD$, so $\triangle BEF \sim \triangle BDC$. Also, $\triangle BEF$ and $\triangle AEC$ are both right-angled and share an angle, so they are also similar. Thus $\triangle AEC \sim \triangle BEF \sim \triangle BDC$, so $\frac{|CE|}{|AC|} = \frac{|CD|}{|BC|}$. 
Since $\triangle ABC$ is given to be isosceles, $|AC| = |BC|$. Therefore, $|CE| = |CD|$.

Also solved by LISA CHEN, student, Moscrop Secondary School, Burnaby, BC; LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; GESINE GEUPEL, student, Max Ernst Gymnasium, Brühl, NRW, Germany; and JUSTINE HANSEN, student, Burnaby North Secondary School, Burnaby, BC.

6. The product of three positive integers is three times as large as their sum. Find all such triples.

Solution by Gesine Geupel, student, Max Ernst Gymnasium, Brühl, NRW, Germany.

Let $a$, $b$, and $c$ denote the three positive integers. Since the order of the integers is not relevant, you may assume that $a \leq b \leq c$. The problem states that $abc = 3(a + b + c)$. Therefore, $ab = 3\left(\frac{a}{c} + \frac{b}{c} + 1\right) \leq 3(1 + 1 + 1) = 9$, because $\frac{a}{c} \leq 1$ and $\frac{b}{c} \leq 1$.

Only a few pairs of integers, $(a, b)$, such that $1 \leq a \leq b$ satisfy that $ab \leq 9$, namely $(1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, 6)$, $(1, 7)$, $(1, 8)$, $(1, 9)$, $(2, 2)$, $(2, 3)$, $(2, 4)$, and $(3, 3)$. Since $abc = 3a + 3b + 3c$, $(ab - 3)c = 3a + 3b$, so $c = \frac{3a + 3b}{ab - 3}$. For each possible pair, $(a, b)$, you may now calculate $c$:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>−3</td>
<td>but $c$ is positive</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>−9</td>
<td>but $c$ is positive</td>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>impossible</td>
<td>2</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>15</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>9</td>
<td>2</td>
<td>4</td>
<td>$\frac{18}{5}$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>7</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>6</td>
<td>but $c \geq b$</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Thus, the only solutions are $(1, 4, 15)$, $(1, 5, 9)$, $(1, 6, 7)$, $(2, 2, 12)$, $(2, 3, 5)$, and $(3, 3, 3)$.

Also solved by LENA CHOI, student, École Dr. Charles Best Secondary School, Coquitlam, BC; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and ROWENA HO, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

This issue’s prize for the best solutions goes to Lena Choi, student, École Dr. Charles Best Secondary School, Coquitlam, BC.

We hope that you will enjoy our featured contest, and we look forward to receiving your solutions at crux-skoliad@cms.math.ca or the postal address listed inside the back cover.